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VOLUME – II

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Untouchability is a crime
Untouchability is inhuman

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PREFACE

This book is designed in the light of the new guidelines and syllabi – 2003 for the Higher Secondary Mathematics, prescribed for the Second Year, by the Government of Tamil Nadu.

The 21st century is an era of Globalisation, and technology occupies the prime position. In this context, writing a text book on Mathematics assumes special significance because of its importance and relevance to Science and Technology.

As such this book is written in tune with the existing international standard and in order to achieve this, the team has exhaustively examined internationally accepted text books which are at present followed in the reputed institutions of academic excellence and hence can be relevant to secondary level students in and around the country.

This text book is presented in two volumes to facilitate the students for easy approach. Volume I consists of Applications of Matrices and Determinants, Vector Algebra, Complex numbers and Analytical Geometry which is dealt with a novel approach. Solving a system of linear equations and the concept of skew lines are new ventures. Volume II includes Differential Calculus – Applications, Integral Calculus and its Applications, Differential Equations, Discrete Mathematics (a new venture) and Probability Distributions.

The chapters dealt with provide a clear understanding, emphasizes an investigative and exploratory approach to teaching and the students to explore and understand for themselves the basic concepts introduced.

Wherever necessary theory is presented precisely in a style tailored to act as a tool for teachers and students.

Applications play a central role and are woven into the development of the subject matter. Practical problems are investigated to act as a catalyst to motivate, to maintain interest and as a basis for developing definitions and procedures.

The solved problems have been very carefully selected to bridge the gap between the exposition in the chapter and the regular exercise set. By doing these exercises and checking the complete solutions provided, students will be able to test or check their comprehension of the material.

Fully in accordance with the current goals in teaching and learning Mathematics, every section in the text book includes worked out and exercise (assignment) problems that encourage geometrical visualisation, investigation, critical thinking, assimilation, writing and verbalization.

We are fully convinced that the exercises give a chance for the students to strengthen various concepts introduced and the theory explained enabling them to think creatively, analyse effectively so that they can face any situation with conviction and courage. In this respect the exercise problems are meant only to students and we hope that this will be an effective tool to develop their talents for greater achievements. Such an effort need to be appreciated by the parents and the well-wishers for the larger interest of the students.

Learned suggestions and constructive criticisms for effective refinement of the book will be appreciated.

K.SRINIVASAN
Chairperson
Writing Team.

SYLLABUS

(1) **APPLICATIONS OF MATRICES AND DETERMINANTS :** ***Adjoint, Inverse*** – Properties, Computation of inverses, solution of system of linear equations by matrix inversion method. ***Rank of a Matrix*** – Elementary transformation on a matrix, consistency of a system of linear equations, Cramer's rule, Non-homogeneous equations, homogeneous linear system, rank method.

(20 periods)

(2) **VECTOR ALGEBRA :** ***Scalar Product*** – Angle between two vectors, properties of scalar product, applications of dot products. ***Vector Product*** – Right handed and left handed systems, properties of vector product, applications of cross product. ***Product of three vectors*** – Scalar triple product, properties of scalar triple product, vector triple product, vector product of four vectors, scalar product of four vectors. ***Lines*** – Equation of a straight line passing through a given point and parallel to a given vector, passing through two given points (derivations are not required). angle between two lines. ***Skew lines*** – Shortest distance between two lines, condition for two lines to intersect, point of intersection, collinearity of three points. ***Planes*** – Equation of a plane (derivations are not required), passing through a given point and perpendicular to a vector, given the distance from the origin and unit normal, passing through a given point and parallel to two given vectors, passing through two given points and parallel to a given vector, passing through three given non-collinear points, passing through the line of intersection of two given planes, the distance between a point and a plane, the plane which contains two given lines, angle between two given planes, angle between a line and a plane. ***Sphere*** – Equation of the sphere (derivations are not required) whose centre and radius are given, equation of a sphere when the extremities of the diameter are given.

(28 periods)

(3) **COMPLEX NUMBERS :** Complex number system, ***Conjugate*** – properties, ordered pair representation. ***Modulus*** – properties, geometrical representation, meaning, polar form, principal value, conjugate, sum, difference, product, quotient, vector interpretation, solutions of polynomial equations, De Moivre's theorem and its applications. ***Roots of a complex number*** – *n*th roots, cube roots, fourth roots. **(20 periods)**

(4) **ANALYTICAL GEOMETRY :** ***Definition of a Conic*** – General equation of a conic, classification with respect to the general equation of a conic, classification of conics with respect to eccentricity. ***Parabola*** – Standard equation of a parabola (derivation and tracing the parabola are not required), other standard parabolas, the process of shifting the origin, general form of the standard equation, some practical problems. ***Ellipse*** – Standard equation of the ellipse (derivation and tracing the ellipse are not required), $x^2/a^2 + y^2/b^2 = 1$, ($a > b$), Other standard form of the ellipse, general forms, some practical problems, ***Hyperbola*** – standard equation (derivation and tracing the hyperbola are not required), $x^2/a^2 - y^2/b^2=1$, Other form of the hyperbola, parametric form of conics, chords. ***Tangents and Normals*** – Cartesian form and ***Parametric form***, equation of chord of contact of tangents from a point (x_1, y_1), ***Asymptotes, Rectangular hyperbola*** – standard equation of a rectangular hyperbola.

(30 periods)

(5) **DIFFERENTIAL CALCULUS – APPLICATIONS I :** ***Derivative as a rate measure*** – rate of change – velocity – acceleration – related rates – Derivative as a measure of slope – tangent, normal and angle between curves. Maxima and Minima. ***Mean value theorem*** – Rolle's Theorem – Lagrange Mean Value Thorem – Taylor's and Maclaurin's series, l' Hôpital's Rule, stationary points – increasing, decreasing, maxima, minima, concavity convexity, points of inflexion.

(28 periods)

(6) **DIFFERENTIAL CALCULUS – APPLICATIONS II : *Errors and approximations*** – absolute, relative, percentage errors, curve tracing, partial derivatives – Euler's theorem. **(10 periods)**

(7) **INTEGRAL CALCULUS AND ITS APPLICATIONS :** Properties of definite integrals, reduction formulae for $\sin^n x$ and $\cos^n x$ (only results), Area, length, volume and surface area **(22 periods)**

(8) **DIFFERENTIAL EQUATIONS :** Formation of differential equations, order and degree, solving differential equations (1^{st} order) – variable separable homogeneous, linear equations. Second order linear equations with constant co-efficients $f(x) = e^{mx}$, $\sin mx$, $\cos mx$, x, x^2. **(18 periods)**

(9A) **DISCRETE MATHEMATICS : *Mathematical Logic*** – Logical statements, connectives, truth tables, Tautologies.

(9B) **GROUPS : *Binary Operations*** – Semi groups – monoids, groups (Problems and simple properties only), order of a group, order of an element. **(18 periods)**

(10) **PROBABILITY DISTRIBUTIONS :** Random Variable, Probability density function, distribution function, mathematical expectation, variance, Discrete Distributions – Binomial, Poisson, Continuous Distribution – Normal distribution

(16 periods)

Total : 210 Periods

CONTENTS

5. DIFFERENTIAL CALCULUS APPLICATIONS - I

5.1 Introduction :

In higher secondary first year we discussed the theoretical aspects of differential calculus, assimilated the process of various techniques involved and created many tools of differentiation. Geometrical and kinematical significances for first and second order derivatives were also interpreted. Now let us learn some practical aspects of differential calculus.

At this level we shall consider problems concerned with the applications to (i) plane geometry, (ii) theory of real functions, (iii) optimisation problems and approximation problems.

5.2 Derivative as a rate measure :

If a quantity y depends on and varies with a quantity x then the rate of change of y with respect to x is $\frac{dy}{dx}$.

Thus for example, the rate of change of pressure p with respect to height h is $\frac{dp}{dh}$. A rate of change with respect to time is usually called as 'the rate of change', the 'with respect to time' being assumed. Thus for example, a rate of change of current 'i' is $\frac{di}{dt}$ and a rate of change of temperature 'θ' is $\frac{d\theta}{dt}$ and so on.

Example 5.1 : The length l metres of a certain metal rod at temperature $\theta°C$ is given by $l = 1 + 0.000050\theta + 0.0000004\theta^2$. Determine the rate of change of length in mm/$°C$ when the temperature is (i) $100°C$ and (ii) $400°C$.

Solution : The rate of change of length means $\frac{dl}{d\theta}$.

$$\text{Since length } l = 1 + 0.000050\theta + 0.0000004\theta^2,$$

$$\frac{dl}{d\theta} = 0.00005 + 0.0000008\theta .$$

(i) when $\theta = 100°C$

$$\frac{dl}{d\theta} = 0.00005 + (0.0000008)(100)$$

$$= 0.00013 \text{ m}/°C = 0.13 \text{ mm}/°C$$

(ii) when $\theta = 400°C$

$$\frac{dl}{d\theta} = 0.00005 + (0.0000008)(400)$$

$$= 0.00037 \text{ m}/°C = 0.37 \text{ mm}/°C$$

Example 5.2 : The luminous intensity I candelas of a lamp at varying voltage V is given by : $I = 4 \times 10^{-4}V^2$. Determine the voltage at which the light is increasing at a rate of 0.6 candelas per volt.

Solution : The rate of change of light with respect to voltage is given by $\frac{dI}{dV}$.

$$\text{Since } I = 4 \times 10^{-4}V^2$$

$$\frac{dI}{dV} = 8 \times 10^{-4}V.$$

When the light is increasing at 0.6 candelas per volt then $\frac{dI}{dV} = +0.6$. Therefore we must have $+0.6 = 8 \times 10^{-4}\,V$, from which,

$$\text{Voltage } V = \frac{0.6}{8 \times 10^{-4}} = 0.075 \times 10^4 = 750 \text{ Volts.}$$

Velocity and Acceleration :

A car describes a distance x metres in time t seconds along a straight road. If the velocity v is constant, then $v = \frac{x}{t}$ m/s i.e., the slope (gradient) of the distance/time graph shown in *Fig.5.1* is constant.

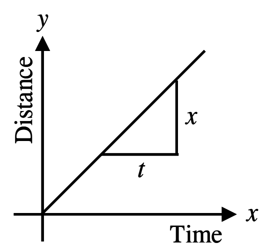


Fig. 5.1

If, however, the velocity of the car is not constant then the distance / time graph will not be a straight line. It may be as shown in *Fig.5.2*

The average velocity over a small time Δt and distance Δx is given by the gradient of the chord AB i.e., the average velocity over time Δt is $\frac{\Delta x}{\Delta t}$.

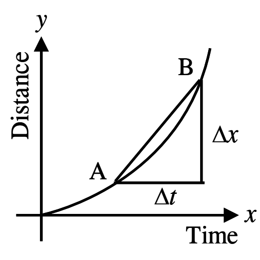


Fig. 5.2

As $\Delta t \to 0$, the chord AB becomes a tangent, such that at point A the velocity is given by $v = \frac{dx}{dt}$. Hence the velocity of the car at any instant is given by gradient of the distance / time graph. If an expression for the distance x is known in terms of time, then the velocity is obtained by differentiating the expression.

The acceleration 'a' of the car is defined as the rate of change of velocity. A velocity / time graph is shown in *Fig.5.3*. If Δv is the change in v and Δt is the corresponding change in time, then $a = \frac{\Delta v}{\Delta t}$. As $\Delta t \to 0$ the chord CD becomes a tangent such that at the point C,

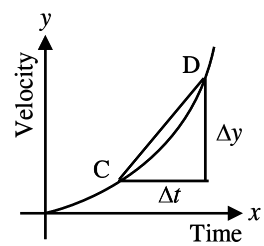


Fig. 5.3

the acceleration is given by $a = \frac{dv}{dt}$

Hence the acceleration of the car at any instant is given by the gradient of the velocity / time graph. If an expression for velocity is known in terms of time t, then the acceleration is obtained by differentiating the expression.

$$\text{Acceleration } a = \frac{dv}{dt}, \text{ where } v = \frac{dx}{dt}$$

$$\text{Hence } a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

The acceleration is given by the second differential coefficient of distance x with respect to time t. The above discussion can be summarised as follows. If a body moves a distance x meters in time t seconds then

(i) distance $x = f(t)$.

(ii) velocity $v = f'(t)$ or $\frac{dx}{dt}$, which is the gradient of the distance / time graph.

(iii) Acceleration $a = \frac{dv}{dt} = f''(t)$ or $\frac{d^2x}{dt^2}$, which is the gradient of the velocity / time graph.

Note : (i) Initial velocity means velocity at $t = 0$

(ii) Initial acceleration means acceleration at $t = 0$.

(iii) If the motion is upward, at the maximum height, the velocity is zero.

(iv) If the motion is horizontal, $v = 0$ when the particle comes to rest.

Example 5.3 : The distance x metres described by a car in time t seconds is given by: $x = 3t^3 - 2t^2 + 4t - 1$. Determine the velocity and acceleration when (i) $t = 0$ and (ii) $t = 1.5\ s$

Solution : distance $x = 3t^3 - 2t^2 + 4t - 1$

$$\text{velocity } v = \frac{dx}{dt} = 9t^2 - 4t + 4 \ m/s$$

$$\text{acceleration } a = \frac{d^2x}{dt^2} = 18t - 4 \ m/s^2$$

(i) When time $t = 0$

$$\text{velocity } v = 9(0)^2 - 4(0) + 4 = 4\ m/s$$

$$\text{and acceleration } a = 18(0) - 4 = -4\ m/s^2$$

(ii) when time $t = 1.5\ sec$

$$\text{velocity } v = 9(1.5)^2 - 4(1.5) + 4 = 18.25\ m/sec$$

$$\text{and acceleration } a = 18(1.5) - 4 = 23\ m/sec^2$$

Example 5.4 : Supplies are dropped from an helicopter and distance fallen in time t seconds is given by $x = \frac{1}{2}gt^2$ where $g = 9.8\ m/sec^2$. Determine the velocity and acceleration of the supplies after it has fallen for 2 seconds.

Solution : distance $x = \frac{1}{2}gt^2 = \frac{1}{2}(9.8)t^2 = 4.9t^2\ m$

$$\text{velocity } v = \frac{dx}{dt} = 9.8t\ m/sec$$

$$\text{acceleration } a = \frac{d^2x}{dt^2} = 9.8\ m/sec^2$$

When time $t = 2$ seconds

$$\text{velocity } v = (9.8)(2) = 19.6\ m/sec$$

and acceleration $a = 9.8\ m/sec^2$ which is the acceleration due to gravity.

Example 5.5 : The angular displacement θ radians of a fly wheel varies with time t seconds and follows the equation $\theta = 9t^2 - 2t^3$. Determine

(i) the angular velocity and acceleration of the fly wheel when time $t = 1$ second and

(ii) the time when the angular acceleration is zero.

Solution : (i) angular displacement $\theta = 9t^2 - 2t^3$ radians.

$$\text{angular velocity } \omega = \frac{d\theta}{dt} = 18t - 6t^2\ rad/s$$

When time $t = 1$ second,

$$\omega = 18(1) - 6(1)^2 = 12 \ rad/s$$

$$\text{angular acceleration} = \frac{d^2\theta}{dt^2} = 18 - 12t \ rad/s^2$$

when t = 1, angular acceleration = 6 rad/s^2

(ii) Angular acceleration is zero $\Rightarrow 18 - 12t = 0$, from which $t = 1.5 \ s$

Example 5.6 : A boy, who is standing on a pole of height 14.7 *m* throws a stone vertically upwards. It moves in a vertical line slightly away from the pole and falls on the ground. Its equation of motion in meters and seconds is $x = 9.8\,t - 4.9t^2$ (i) Find the time taken for upward and downward motions. (ii) Also find the maximum height reached by the stone from the ground.

Solution :

(i) $x = 9.8\,t - 4.9\,t^2$

At the maximum height $v = 0$

$$v = \frac{dx}{dt} = 9.8 - 9.8\,t$$

$v = 0 \Rightarrow t = 1$ sec

$\therefore$ The time taken for upward motion is 1 sec. For each position x, there corresponds a time 't'. The ground position is $x = -14.7$, since the top of the pole is taken as $x = 0$.

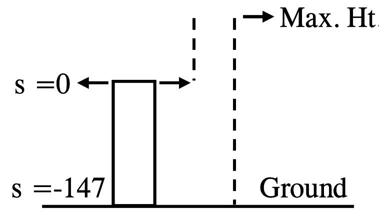


Fig. 5.4

To get the total time, put $x = -14.7$ in the given equation.

i.e., $-14.7 = 9.8\,t - 4.9t^2 \Rightarrow t = -1, 3$

$\Rightarrow t = -1$ is not admissible and hence $t = 3$

The time taken for downward motion is $3 - 1 = 2$ *secs*

(ii) When $t = 1$, the position $x = 9.8(1) - 4.9(1) = 4.9$ *m*

The maximum height reached by the stone = pole height + 4.9 = 19.6 *m*

5.3 Related Rates :

In the related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity. The procedure is to find an equation that relates the two quantities and then use the chain rule to differentiate both sides with respect to time.

We suggest the following problem solving principles that may be followed as a strategy to solve problems considered in this section.

(1) Read the problem carefully.

(2) Draw a diagram if possible.

(3) Introduce notation. Assign symbols to all quantities that are functions of time.

(4) Express the given information and the required rate in terms of derivatives.

(5) Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.

(6) Use the chain rule to differentiate both sides of the equation with respect to t.

(7) Substitute the given information into the resulting equation and solve for the unknown rate.

Illustration : Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm.

Solution :

We start by identifying two things.

(i) The given information : The rate of increase of the volume of air is 100 cm^3/s. and

(ii) The unknown : The rate of increase of the radius when the diameter is 50 cm.

In order to express these quantities mathematically we introduce some suggestive notation.

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that the rates of change are derivatives. In this problem, the volume and the radius are both functions of time t. The rate of increase of the volume with respect to time is the derivative $\frac{dV}{dt}$ and the rate of increase of the radius is $\frac{dr}{dt}$. We can therefore restate the given and the unknown as follows :

Given : $\frac{dV}{dt} = 100\ cm^3/s$ and unknown : $\frac{dr}{dt}$ when $r = 25$ cm.

In order to connect $\frac{dV}{dt}$ and $\frac{dr}{dt}$ we first relate V and r by the formula for the volume of a sphere $V = \frac{4}{3}\pi r^3$.

In order to use the given information, we differentiate both sides of this equation with respect to t. To differentiate the right side, we need to use chain rule as V is a function of r and r is a function of t.

i.e., $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt} = \dfrac{4}{3}\ 3\pi r^2 \dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt}$

Now we solve for the unknown quantity $\dfrac{dr}{dt} = \dfrac{1}{4\pi r^2} \cdot \dfrac{dV}{dt}$

If we put $r = 25$ and $\dfrac{dV}{dt} = 100$ in this equation,

we obtain $\dfrac{dr}{dt} = \dfrac{1 \times 100}{4\pi(25)^2} = \dfrac{1}{25\pi}$

i.e., the radius of the balloon is increasing at the rate of $\dfrac{1}{25\pi}$ *cm/s*.

Example 5.7 : A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 *m/sec* how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the wall ?

Solution : We first draw a diagram and lable it as in *Fig. 5.5*

Let x metres be the distance from the bottom of the ladder to the wall and y metres be the vertical distance from the top of the ladder to the ground. Note that x and y are both functions of time 't'. We are given that $\dfrac{dx}{dt} = 1$ *m/sec* and we are asked to find $\dfrac{dy}{dt}$ when $x = 6$ m.

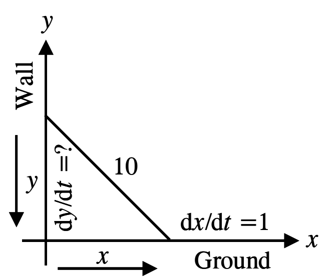


Fig. 5.5

In this question, the relationship between x and y is given by the Pythagoras theorem : $x^2 + y^2 = 100$

Differentiating each side with respect to t, using chain rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the derived rate we obtain,

$$\frac{dy}{dt} = -\frac{x}{y}\ \frac{dx}{dt}$$

When $x = 6$, the Pythagoras theorem gives, $y = 8$ and so substituting these values and $\frac{dx}{dt} = 1$, we get $\frac{dy}{dt} = -\frac{6}{8}(1) = \frac{-3}{4}$ *m/sec*.

The ladder is moving downward at the rate of $\frac{3}{4}$ *m/sec*.

Example 5.8 : A car *A* is travelling from west at 50 km/hr. and car B is travelling towards north at 60 km/hr. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car *A* is 0.3 kilometers and car *B* is 0.4 kilometers from the intersection?

Solution :

We draw *Fig. 5.6* where C is the intersection of the two roads. At a given time t, let *x* be the distance from car *A* to *C*, let y be the distance from car *B* to *C* and let *z* be the distance between the cars *A* and *B* where *x*, *y* and *z* are measured in kilometers.

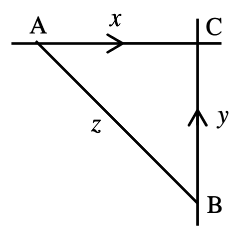


Fig. 5.6

We are given that $\frac{dx}{dt} = -50$ km/hr and $\frac{dy}{dt} = -60$ km/hr.

Note that *x* and *y* are decreasing and hence the negative sign. We are asked to find $\frac{dz}{dt}$. The equation that relate *x*, *y* and *z* is given by the Pythagoras theorem $z^2 = x^2 + y^2$

Differentiating each side with respect to *t*,

we have $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$

When $x = 0.3$ and y = 0.4 km, we get z = 0.5 km and we get

$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)] = -78$ km/hr.

i.e., the cars are approaching each other at a rate of 78 km/hr.

Example 5.9 : A water tank has the shape of an inverted circular cone with base radius 2 metres and height 4 metres. If water is being pumped into the tank at a rate of $2m^3$/min, find the rate at which the water level is rising when the water is 3m deep.

Solution :

We first sketch the cone and label it as in *Fig. 5.7*. Let V, r and h be respectively the volume of the water, the radius of the cone and the height at time t, where t is measured in minutes.

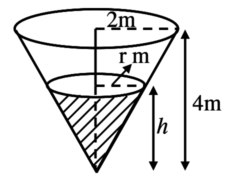


Fig. 5.7

We are given that $\frac{dV}{dt} = 2m^3$/min. and we are asked to find $\frac{dh}{dt}$ when h is 3m.

The quantities V and h are related by the equation $V = \frac{1}{3}\pi r^2 h$. But it is very useful to express V as function of h alone.

In order to eliminate r we use similar triangles in *Fig. 5.7* to write $\frac{r}{h} = \frac{2}{4}$

$\Rightarrow r = \frac{h}{2}$ and the expression for V becomes $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$.

Now we can differentiate each side with respect to t and we have

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3m$ and $\frac{dV}{dt} = 2m^3/min$.

we get, $\frac{dh}{dt} = \frac{4}{\pi(3)^2} . 2 = \frac{8}{9\pi}$ m/min

EXERCISE 5.1

(1) A missile fired from ground level rises x metres vertically upwards in t seconds and $x = 100t - \frac{25}{2}t^2$. Find (i) the initial velocity of the missile, (ii) the time when the height of the missile is a maximum (iii) the maximum height reached and (iv) the velocity with which the missile strikes the ground.

(2) A particle of unit mass moves so that displacement after t secs is given by $x = 3\cos(2t - 4)$. Find the acceleration and kinetic energy at the end of 2 secs. $\left[K.E. = \frac{1}{2}mv^2, m \text{ is mass}\right]$

(3) The distance x metres traveled by a vehicle in time t seconds after the brakes are applied is given by : $x = 20\,t - 5/3t^2$. Determine (i) the speed of the vehicle (in km/hr) at the instant the brakes are applied and (ii) the distance the car travelled before it stops.

(4) Newton's law of cooling is given by $\theta = \theta_0^{\circ} e^{-kt}$, where the excess of temperature at zero time is θ_0°C and at time t seconds is θ°C. Determine the rate of change of temperature after 40 *s*, given that $\theta_0 = 16^{\circ}$ C and $k = -0.03$. [$e^{1.2} = 3.3201$)

(5) The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2.

(6) At noon, ship A is 100 km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4.00 p.m.

(7) Two sides of a triangle are 4m and 5m in length and the angle between them is increasing at a rate of 0.06 rad/sec. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.

(8) Two sides of a triangle have length 12 *m* and 15 *m*. The angle between them is increasing at a rate of 2° /min. How fast is the length of third side increasing when the angle between the sides of fixed length is 60°?

(9) Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarsened such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high ?

5.4 Tangents and Normals (Derivative as a measure of slope)

In this section the applications of derivatives to plane geometry is discussed. For this, let us consider a curve whose equation is $y = f(x)$.

On this curve take a point $P(x_1,y_1)$. Assuming that the tangent at this point is not parallel to the co-ordinate axes, we can write the equation of the tangent line at *P*.

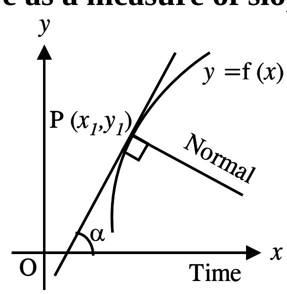


Fig. 5.8

The equation of a straight line with slope (gradient) m passing through (x_1,y_1) is of the form $y - y_1 = m(x - x_1)$. For the tangent line we know the slope $m = f\,'(x_1) = \left(\dfrac{dy}{dx}\right)$ at (x_1,y_1) and so the equation of the tangent is of the form $y - y_1 = f\,'(x_1)\,(x - x_1)$. If m=0, the curve has a horizontal tangent with equation $y = y_1$ at $P(x_1,y_1)$. If $f(x)$ is continuous at $x = x_1$, but $\lim\limits_{x \to x_1} f\,'(x) = \infty \Rightarrow$ the curve has a vertical tangent with equation $x = x_1$.

In addition to the tangent to a curve at a given point, one often has to consider the normal which is defined as follows :

Definition : The normal to a curve at a given point is a straight line passing through the given point, perpendicular to the tangent at this point.

From the definition of a normal it is clear that the slope of the normal m' and that of the tangent m are connected by the equation $m' = -\dfrac{1}{m}$.

$$\text{i.e., } m' = -\frac{1}{f\,'(x_1)} = \frac{-1}{\left(\dfrac{dy}{dx}\right)_{(x_1,y_1)}}$$

Hence the equation of a normal to a curve $y = f(x)$ at a point $P(x_1,y_1)$ is of the form $y - y_1 = -\dfrac{1}{f\,'(x_1)}\,(x - x_1)$.

The equation of the normal at (x_1,y_1) is

(i) $x = x_1$ if the tangent is horizontal (ii) $y = y_1$ if the tangent is vertical and

(iii) $y - y_1 = \dfrac{-1}{m}(x - x_1)$ otherwise.

Example 5.10: Find the equations of the tangent and normal to the curve $y = x^3$ at the point (1,1).

Solution : We have $y = x^3$; slope $y' = 3x^2$.

At the point (1,1), $x = 1$ and $m = 3(1)^2 = 3$.

Therefore equation of the tangent is $y - y_1 = m(x - x_1)$

$y - 1 = 3(x - 1)$ or $y = 3x - 2$

The equation of the normal is $y - y_1 = -\dfrac{1}{m}(x - x_1)$

$y - 1 = \dfrac{-1}{3}(x - 1)$ or $y = -\dfrac{1}{3}x + \dfrac{4}{3}$

Example 5.11 : Find the equations of the tangent and normal to the curve $y = x^2 - x - 2$ at the point (1,– 2).

Solution : We have $y = x^2 - x - 2$; slope, $m = \frac{dy}{dx} = 2x - 1$.

At the point (1,–2), $m = 1$

Hence the equation of the tangent is $y - y_1 = m(x - x_1)$ i.e., $y - (-2) = x - 1$

$$\text{i.e., } y = x - 3$$

Equation of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$$\text{i.e., } y - (-2) = \frac{-1}{1}(x - 1)$$

$$\text{or } y = -x - 1$$

Example 5.12 : Find the equation of the tangent at the point (a,b) to the curve $xy = c^2$.

Solution : The equation of the curve is $xy = c^2$.

Differentiating w.r.to x we get,

$$y + x\frac{dy}{dx} = 0$$

$$\text{or } \frac{dy}{dx} = \frac{-y}{x} \text{ and } m = \left(\frac{dy}{dx}\right)_{(a,\,b)} = \frac{-b}{a}.$$

Hence the required equation of the tangent is

$$y - b = \frac{-b}{a}(x - a)$$

$$\text{i.e., } ay - ab = -bx + ab$$

$$bx + ay = 2ab \text{ or } \frac{x}{a} + \frac{y}{b} = 2$$

Example 5.13 : Find the equations of the tangent and normal at $\theta = \frac{\pi}{2}$ to the curve $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$.

Solution : We have $\frac{dx}{d\theta} = a(1 + \cos\theta) = 2a\cos^2\frac{\theta}{2}$

$$\frac{dy}{d\theta} = -a\sin\theta = -2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\text{Then } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\tan\frac{\theta}{2}$$

$$\therefore \text{ Slope } m = \left(\frac{dy}{dx}\right)_{\theta = \pi/2} = -\tan\frac{\pi}{4} = -1$$

Also for $\theta = \frac{\pi}{2}$, the point on the curve is $\left(a\ \frac{\pi}{2} + a,\ a\right)$.

Hence the equation of the tangent at $\theta = \frac{\pi}{2}$ is

$$y - a = (-1)\left[x - a\left(\frac{\pi}{2} + 1\right)\right]$$

$$\text{i.e., } x + y = \frac{1}{2}\ a\pi + 2a \quad \text{or} \quad x + y - \frac{1}{2}\ a\pi - 2a = 0$$

Equation of the normal at this point is

$$y - a = (1)\left[x - a\left(\frac{\pi}{2} + 1\right)\right]$$

$$\text{or } x - y - \frac{1}{2}\ a\pi = 0$$

Example 5.14 : Find the equations of tangent and normal to the curve $16x^2 + 9y^2 = 144$ at (x_1,y_1) where $x_1 = 2$ and $y_1 > 0$.

Solution : We have $16x^2 + 9y^2 = 144$

(x_1,y_1) lies on this curve, where $x_1 = 2$ and $y_1 > 0$

$\therefore\ (16 \times 4) + 9\,y_1^2 = 144$ or $9\,y_1^2 = 144 - 64 = 80$

$y_1^2 = \frac{80}{9}$ $\therefore\ y_1 = \pm\frac{\sqrt{80}}{3}$. But $y_1 > 0$ $\therefore\ y_1 = \frac{\sqrt{80}}{3}$

$\therefore$ The point of tangency is $(x_1,y_1) = \left(2, \frac{\sqrt{80}}{3}\right)$

We have $16x^2 + 9y^2 = 144$

Differentiating w.r.to x we get $\frac{dy}{dx} = -\frac{32}{18}\,\frac{x}{y} = -\frac{16}{9}\left(\frac{x}{y}\right)$

$$\therefore \text{ The slope at } \left(2, \frac{\sqrt{80}}{3}\right) = \left(\frac{dy}{dx}\right)_{\left(2, \frac{\sqrt{80}}{3}\right)}$$

$$= -\frac{16}{9} \times \frac{2}{\frac{\sqrt{80}}{3}} = -\frac{8}{3\sqrt{5}}$$

$\therefore$ The equation of the tangent is $y - \frac{\sqrt{80}}{3} = -\frac{8}{3\sqrt{5}}(x - 2)$

On simplification we get $8x + 3\sqrt{5}\,y = 36$

Similarly the equation of the normal can be found as $9\sqrt{5}\,x - 24\,y + 14\sqrt{5} = 0$

Example 5.15 : Find the equations of the tangent and normal to the ellipse $x = a\cos\theta,\ y = b\sin\theta$ at the point $\theta = \frac{\pi}{4}$.

Solution : At $\theta = \frac{\pi}{4}$, $(x_1,y_1) = \left(a\cos\frac{\pi}{4},\ b\sin\frac{\pi}{4}\right) = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

$$\frac{dx}{d\theta} = -a\sin\theta,\quad \frac{dy}{d\theta} = b\cos\theta.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b}{a}\cot\theta$$

$$\Rightarrow\ m = = \frac{-b}{a}\cot\frac{\pi}{4} = \frac{-b}{a}$$

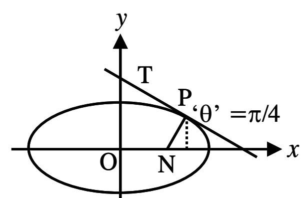


Fig. 5.9

Thus the point of tangency is $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ and the slope is $m = \frac{-b}{a}$.

The equation of the tangent is $y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$ or $bx + ay - ab\sqrt{2} = 0$

The equation of the normal is $y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right)$

$$\text{or}\quad (ax - by)\sqrt{2} - (a^2 - b^2) = 0.$$

Example 5.16 : Find the equation of the tangent to the parabola, $y^2 = 20\,x$ which forms an angle 45° with the x – axis.

Solution : We have $y^2 = 20x$. Let (x_1,y_1) be the tangential point

$$\text{Now } 2yy' = 20 \quad \therefore\ y' = \frac{10}{y} \text{ ie., at } (x_1, y_1) \quad m = \frac{10}{y_1} \qquad \dots (1)$$

But the tangent makes an angle 45° with the x – axis.

$\therefore$ slope of the tangent $m = \tan 45° = 1$ … (2)

From (1) and (2) $\frac{10}{y_1} = 1 \Rightarrow y_1 = 10$

But (x_1,y_1) lies on $y^2 = 20x \Rightarrow y_1^{\,2} = 20\,x_1$

$$100 = 20\,x_1 \text{ or } x_1 = 5$$

$$\text{i.e., } (x_1,y_1) = (5,10)$$

and hence the equation of the tangent at (5, 10) is

$$y - 10 = 1(x - 5)$$

$$\text{or } y = x + 5.$$

Note : This problem is suitable for equation of any tangent to a parabola

$$\text{i.e., } y = mx + \frac{a}{m}$$

5.5 Angle between two curves :

The angle between the curves C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist) Let us represent the two curves C_1 and C_2 by the Cartesian equation $y = f(x)$ and $y = g(x)$ respectively. Let them intersect at $P\ (x_1,y_1)$.

If ψ_1 and ψ_2 are the angles made by the tangents PT_1 and PT_2 to C_1 and C_2 at P, with the positive direction of the x – axis, then $m_1 = \tan \psi_1$ and $m_2 = \tan \psi_2$ are the slopes of PT_1 and PT_2 respectively.

Let ψ be the angle between PT_1 and PT_2. Then $\psi = \psi_2 - \psi_1$ and

$$\tan \psi = \tan (\psi_2 - \psi_1)$$

$$= \frac{\tan \psi_2 - \tan \psi_1}{1 + \tan \psi_1 \tan \psi_2}$$

$$= \frac{m_2 - m_1}{1 + m_1 m_2}$$

where $0 \le \psi < \pi$

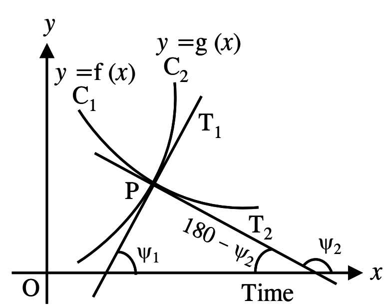


Fig. 5.10

We observe that if their slopes are equal namely $m_1 = m_2$ then the two curves touch each other. If the product $m_1\, m_2 = -1$ then these curves are said to cut at right angles or orthogonally. We caution that if they cut at right angles then $m_1\, m_2$ need not be –1.

Note that in this case ψ_1 is acute and ψ_2 is obtuse and $\psi = \psi_2 - \psi_1$. If ψ_1 is obtuse and ψ_2 is acute, then $\psi = \psi_1 - \psi_2$.

Combining together the angle between tangents can be given as $\psi_1 \sim \psi_2$ or

$$\tan\psi = \tan(\psi_1 \sim \psi_2) = \frac{\tan\psi_1 \sim \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Example 5.17 : Find the angle between the curves $y = x^2$ and $y = (x - 2)^2$ at the point of intersection.

Solution : To get the point of intersection of the curves solve the equation

we get $x^2 = (x - 2)^2$

This gives $x = 1$. When $x = 1$, $y = 1$

$\therefore$ The point of intersection is (1, 1)

Now $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$$\Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(1,1)} = 2$$

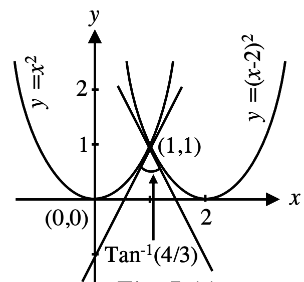


Fig. 5.11

$$y = (x-2)^2 \Rightarrow \frac{dy}{dx} = 2(x-2) \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(1,1)} = -2.$$

If ψ is the angle between them, then

$$\tan\psi = \left|\frac{-2-2}{1-4}\right| = \left|\frac{-4}{-3}\right| \Rightarrow \psi = \tan^{-1}\frac{4}{3}$$

Example 5.18 : Find the condition for the curves

$ax^2 + by^2 = 1$, $a_1x^2 + b_1y^2 = 1$ to intersect orthogonally.

Solution :

If (x_1, y_1) is the point of intersection, then $ax_1^2 + by_1^2 = 1$; $a_1x_1^2 + b_1y_1^2 = 1$

$$\text{then, } x_1^2 = \frac{b_1 - b}{ab_1 - a_1b}, \quad y_1^2 = \frac{a - a_1}{ab_1 - a_1b} \text{ (By Cramer's rule)}$$

$$\text{For } ax^2 + by^2 = 1, \quad m_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-ax_1}{by_1}$$

$$\text{and for } a_1x^2 + b_1y^2 = 1, \quad m_2 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-a_1x_1}{b_1y_1}$$

For orthogonal intersection, we have $m_1m_2 = -1$. This gives

$$\left(\frac{-ax_1}{by_1}\right)\left(\frac{-a_1x_1}{b_1y_1}\right) = -1 \text{ or } \frac{a\,a_1x_1^2}{bb_1y_1^2} = -1.$$

$$aa_1x_1^2 + bb_1y_1^2 = 0 \Rightarrow aa_1\left(\frac{b_1 - b}{ab_1 - a_1b}\right) + bb_1\left(\frac{a - a_1}{ab_1 - a_1b}\right) = 0$$

$$\Rightarrow \quad aa_1(b_1 - b) + bb_1(a - a_1) = 0 \quad \Rightarrow \frac{b_1 - b}{bb_1} + \frac{a - a_1}{aa_1} = 0$$

or $\frac{1}{b} - \frac{1}{b_1} + \frac{1}{a_1} - \frac{1}{a} = 0$ or $\frac{1}{a} - \frac{1}{a_1} = \frac{1}{b} - \frac{1}{b_1}$ which is the required condition.

Example 5.19 : Show that $x^2 - y^2 = a^2$ and $xy = c^2$ cut orthogonally.

Solution : Let (x_1,y_1) be the point of intersection of the given curves

$\therefore\ x_1^2 - y_1^2 = a^2$ and $x_1\,y_1 = c^2$

$$x^2 - y^2 = a^2 \quad \Rightarrow 2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\therefore\ m_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{x_1}{y_1} \quad \text{ie., } m_1 = \frac{x_1}{y_1}$$

$$xy = c^2 \quad \Rightarrow \quad y = \frac{c^2}{x} \quad \Rightarrow \frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$\therefore\ m_2 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-c^2}{x_1^2} \text{ i.e., } m_2 = \frac{-c^2}{x_1^2}$$

$$\therefore\ m_1m_2 = \left(\frac{x_1}{y_1}\right)\left(\frac{-c^2}{x_1^2}\right) = \frac{-c^2}{x_1\,y_1} = \frac{-c^2}{c^2} = -1$$

$\Rightarrow$ the curves cut orthogonally.

Example 5.20 : Prove that the sum of the intercepts on the co-ordinate axes of any tangent to the curve $x = a\cos^4\theta$, $y = a\sin^4\theta$, $0 \le \theta \le \frac{\pi}{2}$ is equal to a.

Solution : Take any point 'θ' as $(a\cos^4\theta, a\sin^4\theta,)$

Now $\frac{dx}{d\theta} = -4a\cos^3\theta\sin\theta$;

and $\frac{dy}{d\theta} = 4a\sin^3\theta\cos\theta$

$\therefore\ \frac{dy}{dx} = -\frac{\sin^2\theta}{\cos^2\theta}$

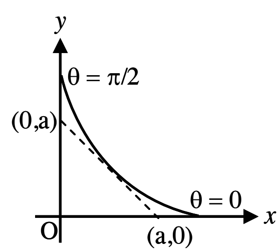


Fig. 5.12

i.e., slope of the tangent at ‘θ’ is $= -\dfrac{\sin^2\theta}{\cos^2\theta}$

Equation of the tangent at ‘θ’ is $(y - a\sin^4\theta) = \dfrac{-\sin^2\theta}{\cos^2\theta}(x - a\cos^4\theta)$

$$\text{or } x\sin^2\theta + y\cos^2\theta = a\sin^2\theta\cos^2\theta$$

$$\Rightarrow \frac{x}{a\cos^2\theta} + \frac{y}{a\sin^2\theta} = 1$$

i.e., sum of the intercepts $= a\cos^2\theta + a\sin^2\theta = a$

EXERCISE 5.2

(1) Find the equation of the tangent and normal to the curves

(i) $y = x^2 - 4x - 5$ at $x = -2$ (ii) $y = x - \sin x\cos x$, at $x = \dfrac{\pi}{2}$

(iii) $y = 2\sin^2 3x$ at $x = \dfrac{\pi}{6}$ (iv) $y = \dfrac{1+\sin x}{\cos x}$ at $x = \dfrac{\pi}{4}$

(2) Find the points on curve $x^2 - y^2 = 2$ at which the slope of the tangent is 2.

(3) Find at what points on the circle $x^2 + y^2 = 13$, the tangent is parallel to the line $2x + 3y = 7$

(4) At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ the tangent is parallel to *(i)* x – axis *(ii)* y – axis.

(5) Find the equations of those tangents to the circle $x^2 + y^2 = 52$, which are parallel to the straight line $2x + 3y = 6$.

(6) Find the equations of normal to $y = x^3 - 3x$ that is parallel to $2x + 18y - 9 = 0$.

(7) Let P be a point on the curve $y = x^3$ and suppose that the tangent line at P intersects the curve again at Q. Prove that the slope at Q is four times the slope at P.

(8) Prove that the curve $2x^2 + 4y^2 = 1$ and $6x^2 - 12y^2 = 1$ cut each other at right angles.

(9) At what angle θ do the curves $y = a^x$ and $y = b^x$ intersect $(a \neq b)$?

(10) Show that the equation of the normal to the curve

$x = a\cos^3\theta$; $y = a\sin^3\theta$ at ‘θ’ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

(11) If the curve $y^2 = x$ and $xy = k$ are orthogonal then prove that $8k^2 = 1$.

5.6 Mean value theorems and their applications :

In this section our main objective is to prove that between any two points of a smooth curve there is a point at which the tangent is parallel to the chord joining two points. To do this we need the following theorem due to Michael Rolle.

5.6.1 Rolle's Theorem : Let f be a real valued function that satisfies the following three conditions :

(i) f is defined and continuous on the closed interval $[a, b]$

(ii) f is differentiable on the open interval (a, b)

(iii) $f(a) = f(b)$

Then there exists atleast one point $c \in (a,b)$ such that $f'(c) = 0$

Some observations :

- Rolle's theorem is applied to the position function $s = f(t)$ of a moving object.
- If the object is in the same place at two different instants $t = a$ and $t = b$ then $f(a) = f(b)$ satisfying hypothesis of Rolle's theorem. Therefore the theorem says that there is some instant of time $t = c$ between a and b where $f'(c) = 0$ i.e., the velocity is 0 at $t = c$.

  Note that this is also true for an object thrown vertically upward (neglecting air resistance).
- Rolle's Theorem applied to theory of equations : If a and b are two roots of a polynomial equation $f(x) = 0$, then Rolle's Theorem says that there is atleast one root c between a and b for $f'(x) = 0$.
- Rolle's theorem implies that a smooth curve cannot intersect a horizontal line twice without having a horizontal tangent in between.
- Rolle's theorem holds trivially for the function $f(x) = c$, where c is a constant on $[a,b]$.
- The converse of Rolle's Theorem is not true *ie.*, if a function f satisfies $f'(c) = 0$ for $c \in (a,b)$ then the conditions of hypothesis need not hold.

Example 5.21 : Using Rolle's theorem find the value(s) of c.

(i) $f(x) = \sqrt{1 - x^2}\,, \quad -1 \leq x \leq 1$

(ii) $f(x) = (x - a)(b - x), \quad a \leq x \leq b,\ a \neq b.$

(iii) $f(x) = 2x^3 - 5x^2 - 4x + 3, \quad \frac{1}{2} \leq x \leq 3$

Solution :

(i) The function is continuous in $[-1,1]$ and differentiable in $(-1,1)$.

$f(1) = f(-1) = 0$ all the three conditions are satisfied.

$$f'(x) = \frac{1}{2}\frac{-2x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$f'(x) = 0 \Rightarrow x = 0.$$

(Note that for $x = 0$, denominator $= 1 \neq 0$) Thus the suitable point for which Rolle's theorem holds is $c = 0$.

(ii) $f(x) = (x - a)(b - x)$, $a \leq x \leq b$, $a \neq b$.

$f(x)$ is continuous on $[a,b]$ and $f'(x)$ exists at every point of (a,b).

$f(a) = f(b) = 0$ All the conditions are satisfied.

$$\therefore f'(x) = (b - x) - (x - a)$$

$$f'(x) = 0 \Rightarrow -2x = -b - a \Rightarrow x = \frac{a+b}{2}$$

The suitable point 'c' of Rolle's theorem is $c = \dfrac{a+b}{2}$

(iii) $f(x) = 2x^3 - 5x^2 - 4x + 3$, $\frac{1}{2} \leq x \leq 3$

f is continuous on $\left[\frac{1}{2}, 3\right]$ and differentiable in $\left(\frac{1}{2}, 3\right)$

$f(½) = 0 = f(3)$. All the conditions are satisfied.

$$f'(x) = 6x^2 - 10x - 4$$

$$f'(x) = 0 \Rightarrow 3x^2 - 5x - 2 = 0 \Rightarrow (3x + 1)(x - 2) = 0 \Rightarrow x = -\frac{1}{3} \text{ or } x = 2.$$

$x = -\frac{1}{3}$ does not lie in $\left(\frac{1}{2}, 3\right)$ $\therefore x = 2$ is the suitable 'c' of Rolle's theorem

Remark : Rolle's theorem cannot be applied if any one of the conditions does not hold.

Example 5.22 : Verify Rolle's theorem for the following :

(i) $f(x) = x^3 - 3x + 3 \quad 0 \leq x \leq 1$

(ii) $f(x) = \tan x, \quad 0 \leq x \leq \pi$

(iii) $f(x) = |x|, \ -1 \leq x \leq 1$

(iv) $f(x) = \sin^2 x, \ 0 \leq x \leq \pi$

(v) $f(x) = e^x \sin x, \quad 0 \leq x \leq \pi$

(vi) $f(x) = x(x - 1)(x - 2), \quad 0 \leq x \leq 2$

Solution :

(i) $f(x) = x^3 - 3x + 3 \quad 0 \le x \le 1$

f is continuous on $[0,1]$ and differentiable in $(0,1)$

$f(0) = 3$ and $f(1) = 1 \therefore f(a) \neq f(b)$

$\therefore$ Rolle's theorem, does not hold, since $f(a) = f(b)$ is not satisfied.

Also note that $f'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

There exists no point $c \in (0,1)$ satisfying $f'(c) = 0$.

(ii) $f(x) = \tan x, \quad 0 \le x \le \pi$

$f'(x)$ is not continuous in $[0,\pi]$ as $\tan x$ tends to $+\infty$ at $x = \dfrac{\pi}{2}$,

$\therefore$ Rolles theorem is not applicable.

(iii) $f(x) = |x|, \; -1 \le x \le 1$

f is continuous in $[-1,1]$ but not differentiable in $(-1,1)$ since $f'(0)$ does not exist.

$\therefore$ Rolles theorem is not applicable.

(iv) $f(x) = \sin^2 x, \; 0 \le x \le \pi$

f is continuous in $[0,\pi]$ and differentiable in $(0,\pi)$. $f(0) = f(\pi) = 0$

(ie.,) f satisfies hypothesis of Rolle's theorem.

$f'(x) = 2 \sin x \cos x = \sin 2x$

$f'(c) = 0 \Rightarrow \sin 2c = 0 \Rightarrow 2c = 0, \pi, 2\pi, 3\pi, \ldots \Rightarrow c = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, \ldots$

since $c = \dfrac{\pi}{2} \in (0,\pi)$, the suitable c of Rolle's theorem is $c = \dfrac{\pi}{2}$.

(v) $f(x) = e^x \sin x, \quad 0 \le x \le \pi$

e^x and $\sin x$ are continuous for all x, therefore the product $e^x \sin x$ is continuous in $0 \le x \le \pi$.

$f'(x) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$ exist in $0 < x < \pi$

$\Rightarrow f'(x)$ is differentiable in $(0,\pi)$.

$$f(0) = e^0 \sin 0 = 0$$

$$f(\pi) = e^{\pi} \sin \pi = 0$$

$\therefore$ f satisfies hypothesis of Rolle's theorem

Thus there exists $c \in (0, \pi)$ satisfying $f'(c) = 0 \Rightarrow e^c(\sin c + \cos c) = 0$

$\Rightarrow e^c = 0$ or $\sin c + \cos c = 0$

$e^c = 0 \Rightarrow c = -\infty$ which is not meaningful here.

$\Rightarrow \sin c = -\cos c \Rightarrow \dfrac{\sin c}{\cos c} = -1 \Rightarrow \tan c = -1 = \tan\dfrac{3\pi}{4}$

$\Rightarrow c = \dfrac{3\pi}{4}$ is the required point.

(vi) $f(x) = x(x-1)(x-2), \quad 0 \le x \le 2,$

f is continuous in $[0,2]$ and differentiable in $(0,2)$

$f(0) = 0 = f(2)$, satisfying hypothesis of Rolle's theorem

$$\text{Now } f'(x) = (x-1)(x-2) + x(x-2) + x(x-1) = 0$$

$$\Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{\sqrt{3}}$$

The required c in Rolle's theorem is $1 \pm \dfrac{1}{\sqrt{3}} \in (0,2)$

Note : There could exist more than one such 'c' appearing in the statement of Rolle's theorem.

Example 5.23 : Apply Rolle's theorem to find points on curve $y = -1 + \cos x$, where the tangent is parallel to x-axis in $[0, 2\pi]$.

Solution :

$f(x)$ is continuous in $[0,2\pi]$ and differentiable in $(0,2\pi)$

$f(0) = 0 = f(2\pi)$ satisfying hypothesis of Rolle's theorem.

Now $f'(x) = -\sin x = 0 \Rightarrow \sin x = 0$

$x = 0, \pi, 2\pi, \ldots$

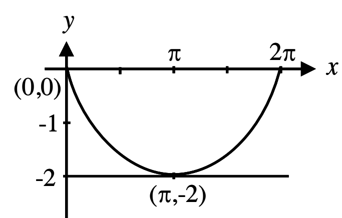


Fig. 5.13

$x = \pi$, is the required c in $(0,2\pi)$. At $x = \pi$, $y = -1 + \cos\pi = -2$.

$\Rightarrow$ the point $(\pi,-2)$ is such that at this point the tangent to the curve is parallel to x-axis.

EXERCISE 5.3

(1) Verify Rolle's theorem for the following functions :

(i) $f(x) = \sin x, \quad 0 \le x \le \pi$

(ii) $f(x) = x^2, \quad 0 \le x \le 1$

(iii) $f(x) = |x-1|, \quad 0 \le x \le 2$

(iv) $f(x) = 4x^3 - 9x, \quad -\dfrac{3}{2} \le x \le \dfrac{3}{2}$

(2) Using Rolle's theorem find the points on the curve $y = x^2+1$, $-2 \le x \le 2$ where the tangent is parallel to x – axis.

5.6.2 Mean Value Theorem (Law of the mean due to Lagrange) :

Many results in this section depend on one central fact called law of the mean or mean value theorem due to Joseph – Louis Lagrange.

Theorem : Let $f(x)$ be a real valued function that satisfies the following conditions :

(i) $f(x)$ is continuous on the closed interval $[a,b]$

(ii) $f(x)$ is differentiable on the open interval (a,b)

Then there exists at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \ldots(1)$$

Some Observations :

- Note that if $f(a) = f(b)$ then the law of the mean reduces to the Rolle's theorem.
- Interpretation of law of the mean when applied to an equation of motion $s = f(t)$:

The quantity $\Delta s = f(b) - f(a)$ is the change in s corresponding to $\Delta t = b - a$ and R.H.S. of (1) is

$$\frac{f(b) - f(a)}{b - a} = \frac{\Delta s}{\Delta t} = \text{average velocity from } t = a \text{ to } t = b.$$

The equation then tells us that there is an instant 'c' between a and b at which the instantaneous velocity $f'(c)$ is equal to the average velocity. For example, if a car has traveled 180 kms in 2 hours then the speedometer must have read 90 kms/hr at least once.

- The slope $f'(c)$ of the curve at $C\,(c, f(c))$ is the same as the slope $\dfrac{f(b) - f(a)}{b - a}$ of the chord joining the points A $(a, f(a))$ and B $(b, f(b))$. Geometrically means that if the function f is continuous on $[a,b]$ and differentiable on (a,b) then there is atleast one number c in (a,b) where the tangent to the curve is parallel to the chord through A and B.

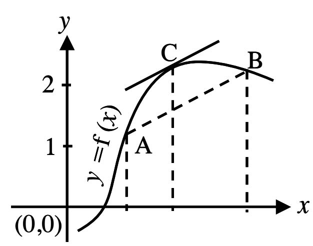


Fig. 5.14

Remarks (1) : Since the value of c satisfies the condition $a < c < b$, it follows that $(c - a) < (b - a)$ or $\frac{c-a}{b-a}$ $(< 1) = \theta$, (say).

$$\text{i.e., } \frac{c-a}{b-a} = \theta \Rightarrow c - a = \theta\,(b - a),\ 0 < \theta < 1.$$

$$\text{But then } \quad c = a + \theta\,(b - a)$$

$\therefore$ the law of the mean can be put in the form

$$\begin{aligned} f(b) - f(a) &= (b - a)\, f'(c) \\ &= (b - a)\, f'[a + \theta\,(b - a)],\ 0 < \theta < 1 \end{aligned}$$

and this is used in calculating approximate values of functions.

(2) Letting $b - a = h$, the above result can be written as

$$f(a + h) = f(a) + hf'(a + \theta h),\ 0 < \theta < 1$$

(3) If we let $a = x$, $h = \Delta x$, law of the mean becomes

$$f(x + \Delta x) = f(x) + \Delta x\, f'(x + \theta\Delta x) \text{ for some } \theta \text{ such that } 0 < \theta < 1.$$

Example 5.24 : Verify Lagrange's law of the mean for $f(x) = x^3$ on $[-2,2]$

Solution : f is a polynomial, hence continuous and differentiable on $[-2, 2]$.

$$f(2) = 2^3 = 8\ ;\ f(-2) = (-2)^3 = -8$$

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$$

By law of the mean there exists an element $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 = \frac{8 - (-8)}{4} = 4$$

$$\text{i.e., } \quad c^2 = \frac{4}{3} \Rightarrow c = \pm\frac{2}{\sqrt{3}}$$

The required 'c' in the law of mean are $\frac{2}{\sqrt{3}}$ and $-\frac{2}{\sqrt{3}}$ as both lie in $[-2,2]$.

Example 5.25 :

A cylindrical hole 4 mm in diameter and 12 mm deep in a metal block is rebored to increase the diameter to 4.12 mm. Estimate the amount of metal removed.

Solution : The volume of cylindrical hole of radius x *mm* and depth 12 *mm* is given by

$$V = f(x) = 12\,\pi x^2$$

$$\Rightarrow\; f'(c) = 24\pi c.$$

To estimate $f(2.06) - f(2)$:

By law of mean,

$$f(2.06) - f(2) = 0.06\; f'(c)$$

$$= 0.06\;(24\,\pi c),\;\; 2 < c < 2.06$$

Take $c = 2.01$

$$f(2.06) - f(2) = 0.06 \times 24\,\pi \times 2.01$$

$$= 2.89\,\pi \text{ cubic mm.}$$

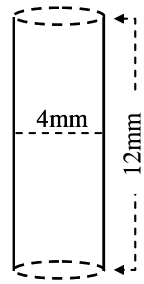


Fig. 5.15

Note : Any suitable c between 2 and 2.06 other than 2.01 also will give other estimates.

Example 5.26 : Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x, how large can $f(2)$ possibly be?

Solution : Since by hypothesis f is differentiable, f is continuous everywhere. We can apply Lagrange's Law of the mean on the interval [0,2]. There exist atleast one 'c'$\in$(0, 2) such that

$$f(2) - f(0) = f'(c)\;(2-0)$$

$$\text{f}(2) = f(0) + 2\,f'(c)$$

$$= -3 + 2\,f'(c)$$

Given that $f'(x) \le 5$ for all x. In particular we know that $f'(c) \le 5$. Multiplying both sides of the inequality by 2, we have

$$2f'(c) \le 10$$

$$f(2) = -3 + 2\,f'(c) \le -3 + 10 = 7$$

i.e., the largest possible value of $f(2)$ is 7.

Example 5.27 : It took 14 sec for a thermometer to rise from $-19°C$ to $100°C$ when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at exactly $8.5°C$/sec.

Solution : Let T be the temperature reading shown in the thermometer at any time t. Then T is a function of time t. Since the temperature rise is continuous and since there is a continuous change in the temperature the function is differentiable too. $\therefore$ By law of the mean there exists a 't_0' in (0, 14) such that

$$\frac{T(t_2) - T(t_1)}{t_2 - t_1} = T'(t_0)$$

Here $T'(t_0)$ is the rate of rise of temperature at C.

Here $t_2 - t_1 = 14$, $T(t_2) = 100$; $T(t_1) = -19$

$$T'(t_0) = \frac{100 + 19}{14} = \frac{119}{14} = 8.5C/\text{sec}$$

EXERCISE 5.4

(1) Verify Lagrange's law of mean for the following functions :

(i) $f(x) = 1 - x^2$, [0,3]

(ii) $f(x) = \frac{1}{x}$, [1,2]

(iii) $f(x) = 2x^3 + x^2 - x - 1$, [0,2]

(iv) $f(x) = x^{2/3}$, [−2,2]

(v) $f(x) = x^3 - 5x^2 - 3x$, [1,3]

(2) If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$ how small can $f(4)$ possibly be?

(3) At 2.00 p.m a car's speedometer reads 30 miles/hr., at 2.10 pm it reads 50 miles / hr. Show that sometime between 2.00 and 2.10 the acceleration is exactly 120 miles /hr^2.

Generalised Law of the Mean :

If $f(x)$ and $g(x)$ are continuous real valued functions on $[a,b]$ and f and g are differentiable on (a,b) with $g'(x) \neq 0$ everywhere on (a,b) then there exist atleast one value of x, say $x = c$, between a and b such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

Remarks :

(1) This theorem is also known as Cauchy's generalised law of the mean.

(2) Lagrange's law of the mean is a particular case of Cauchy law of the mean for the case $g(x) = x$ for all $x \in [a,b]$

(3) Note that $g(b) \neq g(a)$, for, suppose $g(b) = g(a)$, then by Rolle's theorem, $g'(x) = 0$ for some x in (a,b) contradicting hypothesis of the generalized law of the mean.

Extended Law of the mean :

If $f(x)$ and its first $(n - 1)$ derivatives are continuous on $[a,b]$ and if $f^{(n)}(x)$ exists in (a,b), then there exist atleast one value of x, $x = c$ say, in (a,b) such that

$$f(b)=f(a)+\frac{f'(a)}{1!}(b-a)+\frac{f''(a)}{2!}(b-a)^2+\ldots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\frac{f^{(n)}(c)}{n!}(b-a)^n \ldots(1)$$

Remarks : (1) If in the extended law of the mean $b - a = h$ then $b = a + h$ and (1) becomes

$$f(a+h) = f(a) + \frac{f'(a)}{1!} h + \frac{f''(a)}{2!} h^2 + \ldots \frac{f^{(n-1)}(a)}{(n-1)!} h^{n-1} + \frac{f^{(n)}(c)}{n!} h^n \quad \ldots(2)$$

for some $c \in (a, a + h)$ and this is known as Taylor's theorem.

(2) When b is replaced by the variable x then (1) becomes

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \ldots \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

for some $c \in (a, x)$

(3) If n becomes sufficiently large (i.e., ; as $n \to \infty$) in Taylors theorem, then (2) becomes

$$f(a+h) = f(a) + \frac{f'(a)}{1!} h + \frac{f''(a)}{2!} h^2 + \ldots + \frac{f^{(n)}(a)}{n!} h^n + \ldots \quad \ldots(3)$$

provided f is differentiable any number of times. This series of expansion of $f(a + h)$ about the point a is usually known as Taylor's Series.

(4) If in the extended law of the mean a is replaced by 0 and b is replaced with the variable x, (1) becomes,

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} + \frac{f^{(n)}(c)}{n!} x^n \quad \_\_\_(4)$$

for some $c \in (0,x)$ and is known as Maclaurin's theorem.

(5) If n is sufficiently large (*i.e.*, $n \to \infty$) in Maclaurin's theorem, then it becomes $f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \ldots$

provided f is differentiable any number of times, This series expansion of $f(x)$ about the point 0 is usually known as Maclaurin's Series.

Illustration : The Taylor's series expansion of $f(x) = \sin x$ about $x = \frac{\pi}{2}$ is obtained by the following way.

$$f(x) = \sin x \quad ; \quad f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$f'(x) = \cos x \quad ; \quad f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$f''(x) = -\sin x \quad ; \quad f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos x \quad ; \quad f'''\left(\frac{\pi}{2}\right) = 0$$

$$\therefore\ f(x) = \sin x = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x - \frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x - \frac{\pi}{2}\right)^2 + \ldots$$

$$= 1 + 0\left(x - \frac{\pi}{2}\right) + \frac{(-1)}{2!}\left(x - \frac{\pi}{2}\right)^2 + \ldots$$

$$\sin x = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \ldots$$

Example 5.28 :

Obtain the Maclaurin's Series for

1) e^x 2) $\log_e(1 + x)$ 3) arc tan x or $\tan^{-1}x$

Solution :

(1) $f(x) = e^x$; $f(0) = e^0 = 1$

$f'(x) = e^x$; $f'(0) = 1$

$f''(x) = e^x$; $f''(0) = 1$

$\vdots$

$$f(x) = e^x = 1 + \frac{1 . x}{1!} + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \ldots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ holds for all } x$$

(2) $f(x) = \log_e(1 + x)$: $f(0) = \log_e 1 = 0$

$f'(x) = \dfrac{1}{1 + x}$; $f'(0) = 1$

$f''(x) = \dfrac{-1}{(1 + x)^2}$; $f''(0) = -1$

$f'''(x) = \dfrac{+1.2}{(1 + x)^3}$; $f'''(0) = 2!$

$f''''(x) = \dfrac{-1.2.3}{(1 + x)^4}$; $f''''(0) = -(3!)$

$$f(x) = \log_e(1 + x) = 0 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 - \ldots + \ldots.$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots. \quad -1 < x \le 1.$$

(3) $\quad f(x) = \tan^{-1}x \quad ; \quad f(0) = 0$

$$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \ldots ; \; f'(0) = 1 = 1!$$

$$f''(x) = -2x + 4x^3 - 6x^5 \ldots ; \; f''(0) = 0$$

$$f'''(x) = -2 + 12x^2 - 30x^4 \ldots ; \; f'''(0) = -2 = -(2!)$$

$$f^{iv}(x) = 24x - 120x^3 \ldots ; \; f^{iv}(0) = 0$$

$$f^{v}(x) = 24 - 360x^2 \ldots ; \; f^{v}(0) = 24 = 4!$$

$$\tan^{-1}x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{4!}{5!}x^5 + \ldots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$$

holds in $|x| \le 1$.

EXERCISE 5.5

Obtain the Maclaurin's Series expansion for :

(1) e^{2x} (2) $\cos^2 x$ (3) $\frac{1}{1+x}$ (4) $\tan x, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$

5.7 Evaluating Indeterminate forms :

Suppose $f(x)$ and $g(x)$ are defined on some interval $[a,b]$, satisfying Cauchy's generalized law of the mean and vanish at a point $x = a$ of this interval such that $f(a) = 0$ and $g(a) = 0$, then the ratio $\frac{f(x)}{g(x)}$ is not defined for $x = a$ and gives a meaningless expression $\frac{0}{0}$ but has a very definite meaning for values of $x \neq a$. Evaluating the limit $x \to a$ of this ratio is known as evaluating indeterminate forms of the type $\frac{0}{0}$.

If $f(x) = 3x - 2$ and $g(x) = 9x + 7$, then $\frac{3x-2}{9x+7}$ is an indeterminate form of the type $\frac{\infty}{\infty}$ as the numerator and denominator becomes ∞ in the limiting case, x tends to ∞.

We also have other limits $\lim\limits_{x \to \infty} \dfrac{e^x}{x}$, $\lim\limits_{x \to \infty} (x - e^x)$, $\lim\limits_{x \to 0} x^x$, $\lim\limits_{x \to \infty} x^{1/x}$ and $\lim\limits_{x \to 1} x^{1/(x-1)}$ which lead to other indeterminate forms of the types $0 \,.\, \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞ respectively. These symbols must not be taken literally. They are only convenient labels for distinguishing types of behaviour at certain limits. To deal with such indeterminate forms we need a tool that facilitates the evaluation. This tool was devised by John Bernoulli for calculating the limit of a fraction whose numerator and denominator approach zero. This tool today is known as *l*'Hôpital's rule after Guillaume Francois Antoinede *l*'Hôpital.

***l*'Hôpital's rule :**

Let f and g be continuous real valued functions defined on the closed interval $[a,b]$, f, g be differentiable on (a,b) and $g'(c) \neq 0$.

Then if $\lim\limits_{x \to c} f(x) = 0$, $\lim\limits_{x \to c} g(x) = 0$ and if $\lim\limits_{x \to c} \dfrac{f'(c)}{g'(c)} = \mathrm{L}$ it follows that $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \mathrm{L}$.

Remarks :

(1) Using *l*'Hôpital's method, evaluation of the limits of indeterminate forms works faster than conventional methods. For instance, consider $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$. This limit we know is 1, which we obtained through geometrical constructions, a laborious method.

But $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \dfrac{\cos x}{1} = \cos 0 = 1$

(2) Note that *l*'Hôpital's rule can be applied only to differentiable functions for which the limits are in the indeterminate form. For, $\lim\limits_{x \to 0} \dfrac{x+1}{x+3}$ is $\dfrac{1}{3}$ while if *l*'Hôpital's rule is applied $\lim\limits_{x \to 0} \dfrac{x+1}{x+3} = \dfrac{1}{1} = 1$. Here $f(x) = x + 1$ $g(x) = x + 3$ are both differentiable but not in the indeterminate form

(3) The conclusion of *l*'Hôpital's rule is unchanged if $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ and replaced by $\lim\limits_{x \to a} f(x) = \pm \infty$ and $\lim\limits_{x \to a} f(x) = \pm \infty$.

(4) All other indeterminate forms mentioned above can also be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by a suitable transformation.

We need the following result in some problems

Composite Function Theorem :

Result : If $\lim\limits_{x \to a} g(x) = b$ and f is continuous at b,

then $\lim\limits_{x \to a} f(g(x)) = f\left(\lim\limits_{x \to a} g(x)\right)$

Example 5.29 : Evaluate : $\lim\limits_{x \to 0} \frac{x}{\tan x}$

Solution : $\lim\limits_{x \to 0} \frac{x}{\tan x}$ is of the type $\frac{0}{0}$.

$$\therefore \lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{1}{\sec^2 x} = \frac{1}{1} = 1$$

Example 5.30 : Find $\lim\limits_{x \to +\infty} \frac{\sin \frac{1}{x}}{\tan^{-1}\frac{1}{x}}$ if exists

Solution : Let $y = \frac{1}{x}$ As $x \to \infty, y \to 0$

$$\lim_{x \to +\infty} \frac{\sin \frac{1}{x}}{\tan^{-1}\frac{1}{x}} = \lim_{y \to 0} \frac{\sin y}{\tan^{-1} y} = \frac{0}{0}$$

$$= \lim_{y \to 0} \left[\frac{\cos y}{\frac{1}{1+y^2}}\right] = \frac{1}{1} = 1$$

Example 5.31 : $\lim\limits_{x \to \pi/2} \frac{\log(\sin x)}{(\pi - 2x)^2}$

Solution : It is of the form $\frac{0}{0}$

$$\lim_{x \to \pi/2} \frac{\log(\sin x)}{(\pi - 2x)^2} = \lim_{x \to \pi/2} \frac{\frac{1}{\sin x}\cos x}{2(\pi - 2x) \times (-2)}$$

$$= \lim_{x \to \pi/2} \frac{\cot x}{-4(\pi - 2x)} = \frac{0}{0}$$

$$= \lim_{x \to \pi/2} \frac{-\operatorname{cosec}^2 x}{-4 \times -2} = \frac{-1}{8}$$

Note that here *l*'Hôpital's rule, applied twice yields the result.

Example 5.32 : Evaluate : $\displaystyle \lim_{x \to \infty} \frac{x^2}{e^x}$

Solution : $\displaystyle \lim_{x \to \infty} \frac{x^2}{e^x}$ is the type $\dfrac{\infty}{\infty}$

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

Example 5.33 : Evaluate : $\displaystyle \lim_{x \to 0} \left(\operatorname{cosec} x - \frac{1}{x}\right)$

Solution : $\displaystyle \lim_{x \to 0} \left(\operatorname{cosec} x - \frac{1}{x}\right)$ is of the type $\infty - \infty$.

$$\lim_{x \to 0} \left(\operatorname{cosec} x - \frac{1}{x}\right) = \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \left(= \frac{0}{0} \text{ type}\right) = \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \sin x}$$

$$= \frac{0}{2} = 0$$

Example 5.34 : Evaluate : $\displaystyle \lim_{x \to 0} (\cot x)^{\sin x}$

Solution : $\displaystyle \lim_{x \to 0} (\cot x)^{\sin x}$ is of the type ∞^0.

$$\text{Let } y = (\cot x)^{\sin x} \Rightarrow \log y = \sin x \log (\cot x)$$

$$\lim_{x \to 0} (\log y) = \lim_{x \to 0} \sin x \log (\cot x)$$

$$= \lim_{x \to 0} \frac{\log (\cot x)}{\operatorname{cosec} x} \text{ is of the type } \frac{\infty}{\infty}$$

Applying *l*'Hôpital's rule,

$$\lim_{x\to 0} \frac{\log(\cot x)}{\operatorname{cosec} x} = \lim_{x\to 0} \frac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x\to 0} \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \frac{0}{1} = 0$$

i.e., $\lim\limits_{x\to 0} \log y = 0$

By Composite Function Theorem, we have

$$0 = \lim_{x\to 0} \log y = \log\left(\lim_{x\to 0} y\right) \Rightarrow \lim_{x\to 0} y = e^0 = 1$$

Caution : When the existence of $\lim\limits_{x\to a} f(x)$ is not known, $\log\left\{\lim\limits_{x\to a} f(x)\right\}$ is meaningless.

Example 5.35 : Evaluate $\lim\limits_{x\to 0+} x^{\sin x}$

Solution : $\lim\limits_{x\to 0+} x^{\sin x}$ is of the form 0^0.

$$\text{Let } y = x^{\sin x} \Rightarrow \log y = \sin x \log x.$$

Note that x approaches 0 from the right so that $\log x$ is meaningful

$$\text{i.e., } \log y = \frac{\log x}{\operatorname{cosec} x}$$

$$\lim_{x\to 0+} \log y = \lim_{x\to 0+} \frac{\log x}{\operatorname{cosec} x} \text{ which is of the type } \frac{-\infty}{\infty}.$$

Applying l'Hôpital's rule,

$$\lim_{x\to 0+} \frac{\log x}{\operatorname{cosec} x} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x\to 0+} \frac{-\sin^2 x}{x \cos x} \left(\text{of the type } \frac{0}{0}\right)$$

$$= \lim_{x \to 0+} \frac{2 \sin x \cos x}{x \sin x - \cos x} = 0$$

ie., $\lim_{x \to 0+} \log y = 0$

By Composite Function Theorem, we have

$$0 = \lim_{x \to 0+} \log y = \log \lim_{x \to 0+} y \Rightarrow \lim_{x \to 0+} y = e^0 = 1$$

Example 5.36 :

The current at time t in a coil with resistance R, inductance L and subjected to a constant electromotive force E is given by $i = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$. Obtain a suitable formula to be used when R is very small.

Solution :

$$\lim_{R \to 0} i = \lim_{R \to 0} \frac{E\left(1 - e^{\frac{-Rt}{L}}\right)}{R} \quad \text{(is of the type } \frac{0}{0}.\text{)}$$

$$= \lim_{R \to 0} \frac{E \times \frac{t}{L} \; e^{\frac{-Rt}{L}}}{1} = \frac{Et}{L} \Rightarrow \lim_{R \to 0} i = \frac{Et}{L} \text{ is the suitable formula.}$$

EXERCISE 5.6

Evaluate the limit for the following if exists,

(1) $\lim_{x \to 2} \frac{\sin \pi x}{2 - x}$

(2) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(3) $\lim_{x \to 0} \frac{\sin^{-1} x}{x}$

(4) $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2}$

(5) $\lim_{x \to \infty} \frac{\sin \frac{2}{x}}{1/x}$

(6) $\lim_{x \to \infty} \frac{\frac{1}{x^2} - 2 \tan^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}}$

(7) $\lim_{x \to \infty} \frac{\log_e x}{x}$

(8) $\lim_{x \to 0} \frac{\cot x}{\cot 2x}$

(9) $\lim_{x \to 0+} x^2 \log_e x.$

(10) $\lim_{x \to 1} x^{\frac{1}{x-1}}$

(11) $\lim\limits_{x \to \pi/2^-} (\tan x)^{\cos x}$ (12) $\lim\limits_{x \to 0+} x^x$

(13) $\lim\limits_{x \to 0} (\cos x)^{1/x}$

5.8 Monotonic Functions :

Increasing, Decreasing Functions

Differential calculus has varied applications. We have already seen some applications to geometrical, physical and practical problems in sections 5.2, 5.3 and 5.4 In this section, we shall study some applications to the theory of real functions.

In sketching the graph of a function it is very useful to know where it raises and where it falls. The graph shown in *F*ig. 5.16 raises from *A* to *B*, falls from *B* to *C*, and raises again from *C* to *D*.

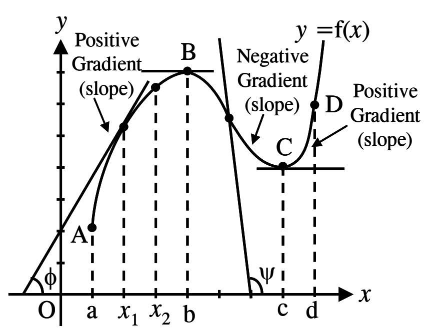


Fig. 5.16

The function f is said to be increasing on the interval $[a,b]$, decreasing on $[b,c]$, and increasing again on $[c,d]$. We use this as the defining property of an increasing function.

Definition : A function f is called increasing on an interval I if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ in I. It is called decreasing on I if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$ in I.

A function that is completely increasing or completely decreasing on I is called monotonic on I.

In the first case the function f preserves the order.

i.e., $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ and in the later case the function f reverses the order *i.e.*, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$. Thanks to the order preserving property, increasing functions are also known as order preserving functions. Similarly, the decreasing functions are also known as order reversing functions.

Illustrations :

(i) Every constant function is an increasing function.

(ii) Every identity function is an increasing function.

(iii) The function $f(x) = \sin x$ is not an increasing function on R; but $f(x) = \sin x$ is increasing on $\left[0, \frac{\pi}{2}\right]$.

(iv) The function $f(x) = 4 - 2x$ is decreasing

(v) The function $f(x) = \sin x$ is decreasing in the interval $\left[\frac{\pi}{2}, \pi\right]$

Note that f is increasing is equivalent to $(-f)$ is decreasing.

Do you agree that each constant function is both increasing and decreasing?

Caution : It is incorrect to say that if a function is not increasing, then it is decreasing. It may happen that a function is neither increasing nor decreasing. For instance, if we consider the interval $[0,\pi]$, the function $\sin x$ is neither increasing nor decreasing. It is increasing on $\left[0, \frac{\pi}{2}\right]$ and decreasing on $\left[\frac{\pi}{2}, \pi\right]$. There are other functions that are even worse. They are not monotonic on any subinterval also. But most of the functions that we consider are not so bad.

Usually, by looking at the graph of the function one can say whether the function is increasing or decreasing or neither. The graph of an increasing function does not fall as we go from left to right while the graph of a decreasing function does not rise as we go from left to right. But if we are not given the graph, how do we decide whether a given function is monotonic or not ? Theorem 1 gives us a criterion to do just that.

Theorem 1 : Let I be an open interval. Let $f : \mathrm{I} \to \mathrm{R}$ be differentiable. Then

(i) f is increasing if and only if $f'(x) \geq 0$ for all x in I.

(ii) f is decreasing if and only if $f'(x) \leq 0$ for all x in I.

Proof : (i) Let f be increasing and $x \in \mathrm{I}$. Since f is differentiable $f'(x)$ exists and is given by $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. If $h > 0$, then $x + h > x$ and since f is increasing, $f(x + h) \geq f(x)$. Hence $f(x + h) - f(x) \geq 0$.

If $h < 0$, then $x + h < x$ and $f(x + h) \leq f(x)$. Hence $f(x + h) - f(x) \leq 0$

So either $f(x + h) - f(x)$ and h are both non-negative or they are both non – positive.

Therefore $\frac{f(x+h) - f(x)}{h}$ is non-negative for all non-zero values of h and $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ must also be non-negative. Thus, $f'(x) \geq 0$

Conversely, let $f'(x) \geq 0$, for all x in I. Let $x_1 < x_2$ in I. We shall prove that $f(x_1) \leq f(x_2)$.

By the Law of mean, $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$, for $x_1 < c < x_2$

Since, $f'(c) \geq 0$, we have $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$. Also $x_2 - x_1 > 0$ ($\therefore$ $x_1 < x_2$)

Thus $f(x_2) - f(x_1) \geq 0$ or $f(x_1) \leq f(x_2)$. Hence f is increasing

(ii) can be proved in a similar way. It can also be deduced by applying result (i) to the function $(-f)$.

Geometrical interpretation : The above theorem expresses the following geometric fact. If on an interval I = $[a,b]$ a function $f(x)$ increases, then the tangent to the curve $y = f(x)$ at each point on this interval forms an acute angle φ with the x-axis or (at certain points) is horizontal *(See Fig.5.16)*, the tangent of this angle is not negative. Therefore $f'(x) = \tan \varphi \geq 0$. If the function $f(x)$ decreases on the interval $[b,c]$ then the angle of inclination of the tangents form an obtuse angle (or, at some points, the tangent is horizontal) ; the tangent of this angle is not positive $f'(x) = \tan \psi \leq 0$.

From the class of increasing functions we can separate out functions which are strictly increasing. The following definition gives the precise meaning of the term strictly increasing function.

Definition : $f : I \to R$ is said to be strictly increasing if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$. We can similarly say that a function defined on I is strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$

For example, a constant function is not strictly increasing, nor is it strictly decreasing (*Fig. 5.17)*. The greatest integer function $f(x) = \lfloor x \rfloor$ too, is increasing (*Fig. 5.18)*, but not strictly increasing, where as the function $f(x) = x$ is strictly increasing *(Fig. 5.19)*.

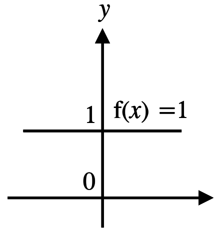


Fig. 5.17

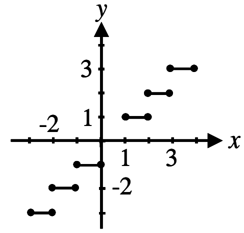


Fig. 5.18

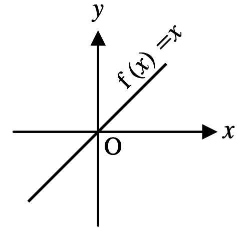


Fig. 5.19

Theorem 2 :

(i) Let f' be positive on I. Then f is strictly increasing on I.

(ii) Let f' be negative on I. Then f is strictly decreasing on I.

The proof of the theorem is easy and is left as an exercise.

Corollary : f is strictly monotonic on the interval I, if f' is of the same sign through out I.

You may have noticed that there is a difference between the statement of Theorem 1 and Theorem 2.

"f is increasing if and only if f' is non – negative"

" If $f' > 0$, then f is strictly increasing".

Can we have if and only if in Theorem 2 also ?

The answer is no as shown in the following example.

Illustration : Define $f : R \to R$ by $f(x) = x^3$.

Suppose $x_1 < x_2$, Then $x_2 - x_1 > 0$ and $x_1^2 + x_2^2 > 0$

$$\begin{aligned} \text{This implies } x_2^3 - x_1^3 &= (x_2 - x_1)(x_2^2 + x_1^2 + x_1 x_2) \\ &= (x_2 - x_1)\frac{1}{2}[(x_1^2 + x_2^2) + (x_1 + x_2)^2] > 0 \\ \Rightarrow x_1^3 &< x_2^3 \end{aligned}$$

Thus whenever $x_1 < x_2$, $f(x_1) < f(x_2)$.

Hence $f(x) = x^3$ is strictly increasing.

But its derivate $f'(x) = 3x^2$ and $f'(0) = 0$.

Hence its derivate f' is not strictly positive.

Note: If a function changes its signs at different points of a region (interval) then the function is not monotonic in that region. So to prove the non- monotonicity of a function, it is enough to prove that f' has different signs at different points.

Example 5.37 : Prove that the function $f(x) = \sin x + \cos 2x$ is not monotonic on the interval $\left[0, \frac{\pi}{4}\right]$.

Solution :

$$\begin{aligned} \text{Let } f(x) &= \sin x + \cos 2x \\ \text{Then } f'(x) &= \cos x - 2\sin 2x \\ \text{Now } f'(0) &= \cos 0 - 2\sin 0 = 1 - 0 = 1 > 0 \\ \text{and } f'\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) - 2\sin 2\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} - 2 \times 1 < 0 \end{aligned}$$

Thus f' is of different signs at 0 and $\frac{\pi}{4}$ Therefore f is not monotonic on $\left[0,\ \frac{\pi}{4}\right]$

Example 5.38 : Find the intervals in which $f(x) = 2x^3 + x^2 - 20x$ is increasing and decreasing.

Solution : $f'(x) = 6x^2 + 2x - 20 = 2(3x^2 + x - 10) = 2(x + 2)(3x - 5)$

Now $f'(x) = 0 \Rightarrow x = -2$, and $x = 5/3$. The values -2 and ${}^5/_3$ divide the real line (the domain of $f(x)$) into intervals $(-\infty, -2)$, $(-2, 5/3)$ and $(5/3, \infty)$.

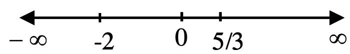


Fig. 5.20

Interval	***x + 2***	***3x – 5***	***f′(x)***	***Interval of inc / dec***
$-\infty < x < -2$	–	–	+	Increasing on $(-\infty, -2]$
$-2 < x < 5/3$	+	–	–	decreasing on $[-2, 5/3]$
$5/3 < x < \infty$	+	+	+	increasing on $[5/3, \infty)$

Note (i) : If the critical numbers are not included in the intervals, then the intervals of increasing (decreasing) becomes strictly increasing (strictly decreasing)

Note : (ii) The intervals of *inc* / *dec* can be obtained by taking and checking a sample point in the sub-interval.

Example 5.39 : Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in [0,1]

Solution :
$$f(x) = x^2 - x + 1$$
$$f'(x) = 2x - 1$$

$f'(x) \geq 0$ for $x \geq \frac{1}{2}$ i.e., $x \in \left[\frac{1}{2}, 1\right]$ $\therefore$ $f(x)$ is increasing on $\left[\frac{1}{2}, 1\right]$

Also $f'(x) \leq 0$ for $x \leq \frac{1}{2} \Rightarrow x \in \left[0,\ \frac{1}{2}\right]$. Also $f'(x)$ is decreasing on $\left[0,\ \frac{1}{2}\right]$

Therefore in the entire interval [0,1] the function $f(x)$ is neither increasing nor decreasing.

Example 5.40 : Discuss monotonicity of the function

$f(x) = \sin x,\ x \in [0, 2\pi]$

Solution : $f(x) = \sin x$ and $f'(x) = \cos x = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}$ in $[0, 2\pi]$ Now $f'(x) \geq 0$ for $0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$. Therefore $f(x) = \sin x$ is increasing on $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$ i.e., $\sin x$ is increasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

Also, $f'(x) \leq 0$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$. Therefore $f(x) = \sin x$ is decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Example 5.41 : Determine for which values of x, the function $y = \frac{x-2}{x+1}$, $x \neq -1$ is strictly increasing or strictly decreasing.

Solution :

$$y = \frac{x-2}{x+1},\ x \neq -1 \qquad \frac{dy}{dx} = \frac{(x+1)\,1 - (x-2)\,1}{(x+1)^2} = \frac{3}{(x+1)^2} > 0 \text{ for all } x \neq -1.$$

$\therefore$ y is strictly increasing on $R - \{-1\}$.

Example 5.42 : Determine for which values of x, the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing and for which it is decreasing. Also determine the points where the tangents to the graph of the function are parallel to the x axis.

Solution : $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$

$f'(x) = 0 \Rightarrow x = 2, 3$. Therefore the points 2 and 3 divide the real line into $(-\infty, 2)$, $(2, 3)$ $(3, \infty)$.

Interval	***x – 2***	***x – 3***	***f ′(x)***	***Intervals of inc / dec***
$-\infty < x < 2$	–	–	+	increasing on $(-\infty, 2]$
$2 < x < 3$	+	–	–	decreasing on $[2, 3]$
$3 < x < \infty$	+	+	+	increasing on $[3, \infty)$

The points where the tangent to the graph of the function are parallel to the x – axis are given by $f'(x) = 0$, ie., when $x = 2, 3$ Now $f(2) = 29$ and $f(3) = 28$.

Therefore the required points are (2, 29) and (3, 28)

Example 5.43 :

Show that $f(x) = \tan^{-1}(\sin x + \cos x)$, $x > 0$ is a strictly increasing function in the interval $\left(0, \frac{\pi}{4}\right)$.

Solution : $f(x) = \tan^{-1}(\sin x + \cos x)$.

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} (\cos x - \sin x) = \frac{\cos x - \sin x}{2 + \sin 2x} > 0$$

since $\cos x - \sin x > 0$ in the interval $\left(0, \frac{\pi}{4}\right)$

and $2 + \sin 2x > 0$)

$\therefore$ $f(x)$ is strictly increasing function of x in the interval $\left(0 , \frac{\pi}{4}\right)$

EXERCISE 5.7

(1) Prove that e^x is strictly increasing function on R.

(2) Prove that $\log x$ is strictly increasing function on $(0, \infty)$

(3) Which of the following functions are increasing or decreasing on the interval given ?

(i) $x^2 - 1$ on $[0,2]$ (ii) $2x^2 + 3x$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(iii) e^{-x} on $[0,1]$ (iv) $x(x - 1)(x + 1)$ on $[-2, -1]$

(v) $x \sin x$ on $\left[0, \frac{\pi}{4}\right]$

(4) Prove that the following functions are not monotonic in the intervals given.

(i) $2x^2 + x - 5$ on $[-1,0]$ (ii) $x(x - 1)(x + 1)$ on $[0,2]$

(iii) $x \sin x$ on $[0,\pi]$ (iv) $\tan x + \cot x$ on $\left(0, \frac{\pi}{2}\right)$

(5) Find the intervals on which f is increasing or decreasing.

(i) $f(x) = 20 - x - x^2$ (ii) $f(x) = x^3 - 3x + 1$

(iii) $f(x) = x^3 + x + 1$ (iv) $f(x) = x - 2\sin x$, $[0, 2\pi]$

(v) $f(x) = x + \cos x$ in $[0, \pi]$ (vi) $f(x) = \sin^4 x + \cos^4 x$ in $[0, \pi/2]$

Inequalities :

Example 5.44 :

Prove that $e^x > 1 + x$ for all $x > 0$.

Solution : Let $f(x) = e^x - x - 1 \Rightarrow \quad f'(x) = e^x - 1 > 0$ for $x > 0$

i.e., f is strictly increasing function. $\therefore$ for $x > 0$, $f(x) > f(0)$

i.e., $(e^x - x - 1) > (e^0 - 0 - 1)$; $e^x > x + 1$

Example 5.45 :

Prove that the inequality $(1 + x)^n > 1+nx$ is true whenever $x > 0$ and $n > 1$.

Solution : Consider the difference $f(x) = (1 + x)^n - (1 + nx)$

$$\text{Then } f'(x) = n(1 + x)^{n-1} - n = n[(1 + x)^{n-1} - 1]$$

Since $x > 0$ and $n - 1 > 0$, we have $(1 + x)^{n-1} > 1$, so $f'(x) > 0$.

Therefore f is strictly increasing on $[0, \infty)$.

For $x > 0 \Rightarrow f(x) > f(0)$ i.e., $(1 + x)^n - (1 + nx) > (1 + 0) - (1 + 0)$

i.e., $(1 + x)^n - (1 + nx) > 0$ i.e., $(1 + x)^n > (1 + nx)$

Example 5.46 : Prove that $\sin x < x < \tan x$, $x \in \left(0, \frac{\pi}{2}\right)$

Solution :

Let $f(x) = x - \sin x$

$f'(x) = 1 - \cos x > 0$ for $0 < x < \frac{\pi}{2}$

$\therefore$ f is strictly increasing.

For $x > 0$, $f(x) > f(0)$

$\Rightarrow x - \sin x > 0 \Rightarrow x > \sin x \quad \ldots (1)$

Let $g(x) = \tan x - x$

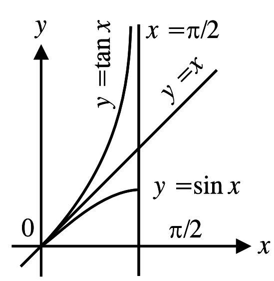


Fig. 5.21

$g'(x) = \sec^2 x - 1 = \tan^2 x > 0$ in $\left(0, \frac{\pi}{2}\right)$

$\therefore$ g is strictly increasing

For $x > 0$, $f(x) > f(0) \Rightarrow \tan x - x > 0 \Rightarrow \tan x > x \ldots$ (2)

From (1) and (2) $\quad \sin x < x < \tan x$

EXERCISE 5.8

(1) Prove the following inequalities :

(i) $\cos x > 1 - \frac{x^2}{2}$, $x > 0$ (ii) $\sin x > x - \frac{x^3}{6}$, $x > 0$

(iii) $\tan^{-1} x < x$ for all $x > 0$ (iv) $\log (1 + x) < x$ for all $x > 0$.

5.9 Maximum and Minimum values and their applications :

"For since the fabric of the Universe is most perfect and the work of a most wise creator, nothing at all takes place in the Universe in which some rule of maximum or minimum does not appear"

Leonard Euler

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. In many cases these problems can be reduced to finding the maximum or minimum values of a function. Many practical problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation.

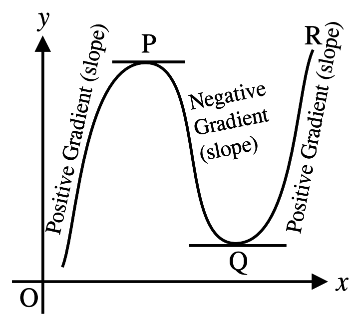


Fig. 5.22

Let us first explain exactly what we mean by maximum and minimum values.

In *fig 5.22* the gradient (rate of change) of the curve changes from positive between O and P to negative between P and Q and positive again between Q and R. At point P, the gradient is zero and as x increases, the gradient (slope) of the curve changes from positive just before P to negative just after. Such a point is called a maximum point and appears as the **'crest of a wave'**.

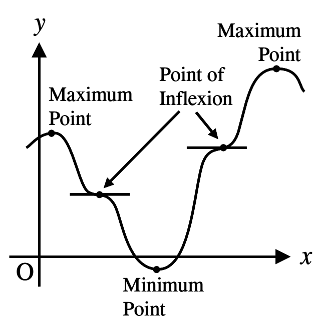


Fig. 5.23

At point Q, the gradient is also zero and as x increases the gradient of the curve changes from negative just before Q to positive just after. Such a point is called a minimum point and appears as **'the bottom of a valley'**. Points such as P and Q are given the general name, turning points.

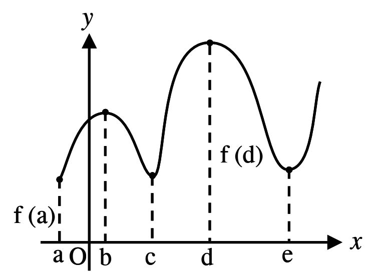


Fig. 5.24

It is possible to have a turning point, the gradient on either side of which is the same. Such a point is given the special name of a point of inflection as shown in *Fig 5.23.*

Definition : A function f has an absolute maximum at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called maximum value of f on D. Similarly f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number $f(c)$ is called the minimum value of f on D. The maximum and minimum values of f are called extreme values of f.:

Fig.5.24 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

In *Fig. 5.24* if we consider only values of x near b, for instance, if we restrict our attention to the interval (a,c) then $f(b)$ is the largest of those values of $f(x)$ and is called a local maximum value of f. Likewise $f(c)$ is called a local minimum value of f because $f(c) \leq f(x)$ for x near c, in the interval (b,d). The function f also has a local minimum at e. In general we have the following definition.

Definition : A function f has a local maximum (or relative maximum) at c if there is an open interval I containing c such that $f(c) \geq f(x)$ for all x in I. Similarly, f has a local minimum at c if there is an open interval I containing c such that $f(c) \leq f(x)$ for all x in I.

Illustrations : (1) The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times since $\cos 2n\pi = 1$ for any integer and $-1 \leq \cos x \leq 1$ for all x. Like wise $\cos (2n + 1)\pi = -1$ is its (local and absolute) minimum value, n is any integer.

(2) If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x. Therefore $f(0) = 0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest on the parabola $y = x^2$ *See Fig.5.25* However, there is no highest point on the parabola and so this function has no maximum value.

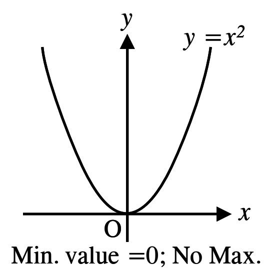


Fig. 5.25

(3) If $f(x) = x^3$ then from the graph of $f(x)$ shown in *Fig 5.26*, we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact it has no local extreme values either.

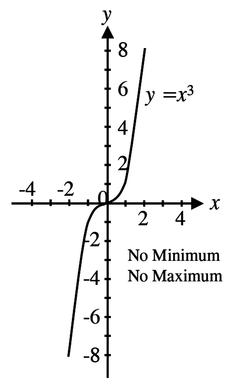


Fig. 5.26

(4) Consider the function

$f(x) = 3x^4 - 16x^3 + 18x^2$; $-1 \leq x \leq 4$. The graph is shown in *Fig. 5.27*.

We can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1)=37$. Also $f(0) = 0$ is a local minimum and $f(3)= -27$ is both local and absolute minimum.

We have seen that some functions have extreme values, while others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

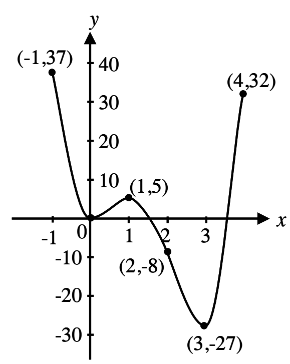


Fig. 5.27

The Extreme value theorem : If f is continuous on a closed interval $[a,b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value f(d) at some number c and d in $[a,b]$

The next two examples show that a function need not possess extreme values if either of the hypotheses (continuity or closed interval) is omitted from the extreme value theorem.

(5) Consider the function

$$f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 0, & 1 \le x \le 2 \end{cases}$$

The function is defined on the closed interval [0,2] but has no maximum value. Notice that the range of f is the interval [0,1). The function takes on value close to 1 but never attains the value 1.

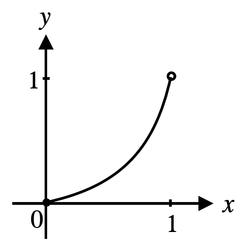


Fig. 5.28

This is because the hypotheses of f to be continuous fails. Note that $x = 1$ is a point of discontinuity, for,

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x^2) = 1 \ ; \ \lim_{x \to 1+} f(x) = 0$$

(6) The function $f(x) = x^2$, $0 < x < 2$ is continuous on the interval (0,2) but has neither a maximum nor a minimum value. The range of f is the interval (0,4). The values 0 and 4 are never taken on by f. This is because the interval (0,2) is not closed.

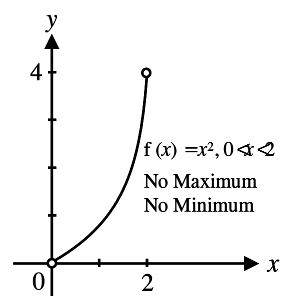


Fig. 5.29

If we alter the function by including either end point of the interval (0,2) then we get one of the situations shown in *Fig. 5.30*, *Fig. 5.31*, *Fig. 5.32* In particular the function $f(x) = x^2$, $0 \le x \le 2$ is continuous on the closed interval [0,2]. So the extreme value theorem says that the function has an absolute maximum and an absolute minimum.

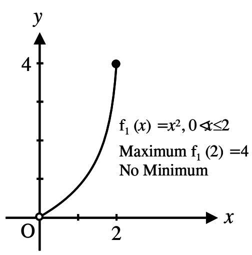


Fig. 5.30

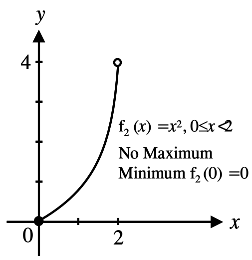


Fig. 5.31

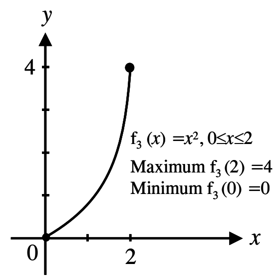


Fig. 5.32

Inspite of the above examples we point out that there are functions which are neither continuous nor differentiable but still attains minimum and maximum values. For instance, consider

$$f(x) = \begin{cases} 1 & , \ x \text{ is irrational} \\ 0 & , \ x \text{ is rational} \end{cases}$$

(This function is known as characteristic function on the set of irrational numbers)

This function is nowhere differentiable and everywhere discontinuous. But the maximum value is 1 and the minimum value is 0.

The extreme value theorem says that a continuous function on a closed interval has a maximum value and minimum value, but it does not tell us how to find their extreme values.

Fig. 5.33 shows the graph of a function f with a local maximum at c and a local minimum at d. It appears that at the maximum and minimum points the tangent line is horizontal and therefore has slope zero. We know that the derivative is the slope of the tangent line, so it appear that $f'(c) = 0$ and $f'(d) = 0$.

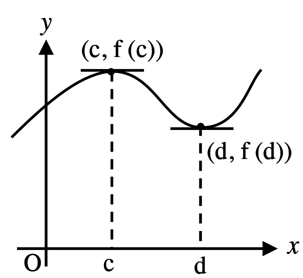


Fig. 5.33

The following theorem shows that this is always true for differentiable functions.

Fermat's Theorem : If f has a local extremum (maximum or minimum) at c and if $f'(c)$ exists then $f'(c) = 0$.

The following examples caution us that we cannot locate extreme values simply by setting $f'(x) = 0$ and solving for x.

(7) The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value cannot be found by setting $f'(x) = 0$ because $f'(x)$ does not exist.

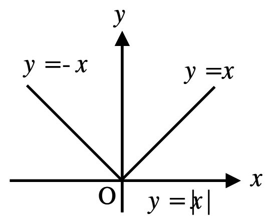


Fig. 5.34

(8) The function $f(x) = 3x - 1, 0 \le x \le 1$ has its maximum value when $x = 1$ but $f'(1) = 3 \neq 0$. This does not contradict Fermat's Theorem. Since $f(1) = 2$ is not a local maximum.

Note that the number 1 is not contained in an open interval in the domain of f.

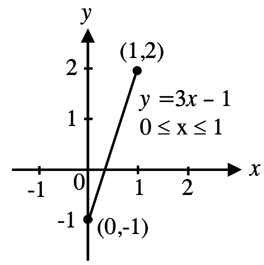


Fig. 5.35

Remark : The above examples demonstrate that even when $f'(c) = 0$ there need not be a maximum or minimum at c. Further more, there may be an extreme value even when $f'(c) \neq 0$ or when $f'(c)$ does not exist.

(9) If $f(x) = x^3$. Then $f'(x) = 3x^2$,

so $f'(0) = 0$.

But f has no maximum or minimum at 0 as you can see from its graph. (observe that $x^3 > 0$ for $x > 0$ and $x^3 < 0$ for $x < 0$).

The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0,0). Instead of having a maximum or minimum at (0, 0) the curve crosses its horizontal tangent there.

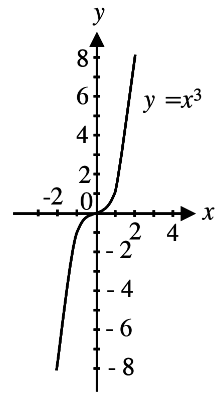


Fig. 5.36

Fermats' theorem does suggest that we should atleast start looking for extreme values of f at the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist.

Definition : A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Stationary points are critical numbers c in the domain of f, for which $f'(c) = 0$.

Example 5.47 : Find the critical numbers of $x^{3/5}(4 - x)$

Solution :

$$f(x) = 4x^{3/5} - x^{8/5}$$

$$f'(x) = \frac{12}{5}x^{-2/5} - \frac{8}{5}x^{3/5}$$

$$= \frac{4}{5}x^{-2/5}(3 - 2x)$$

Therefore $f'(x) = 0$ if $3 - 2x = 0$ *i.e.*, if $x = \frac{3}{2}$. $f'(x)$ does not exist when $x = 0$. Thus the critical numbers are 0 and $\frac{3}{2}$.

Note that if f has a local extremum at c, then c is a critical number of f, but not vice versa.

To find the absolute maximum and absolute minimum values of a continuous function f on a closed interval $[a,b]$:

(1) Find the values of f at the critical numbers, of f in (a,b).

(2) Find the values of $f(a)$ and $f(b)$

(3) The largest of the values from steps 1 and 2 is the absolute maximum value, the smallest of these values is the absolute minimum value.

Example 5.48 : Find the absolute maximum and minimum values of the function. $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \le x \le 4$

Solution : Note that f is continuous on ; $\left[-\frac{1}{2}, 4\right]$

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

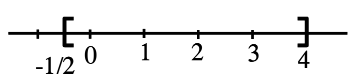


Fig. 5.37

Since $f'(x)$ exists for all x, the only critical numbers of f are $x = 0$, $x = 2$.

Both of these critical numbers lie in the interval $\left[-\frac{1}{2}, 4\right]$. Value of f at these critical numbers are $f(0) = 1$ and $f(2) = -3$.

The values of f at the end points of the interval are

$$f\left(-\tfrac{1}{2}\right) = \left(-\tfrac{1}{2}\right)^3 - 3\left(-\tfrac{1}{2}\right)^2 + 1 = \frac{1}{8}$$

$$\text{and } f(4) = 4^3 - 3 \times 4^2 + 1 = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Note that in this example the absolute maximum occurs at an end point, whereas the absolute minimum occurs at a critical number.

Example 5.48(a): Find the absolute maximum and absolute minimum values of

$f(x) = x - 2\sin x,\ 0 \le x \le 2\pi.$

Solution : $f(x) = x - 2\sin x$, is continuous in $[0, 2\pi]$

$$f'(x) = 1 - 2\cos x$$

$$f'(x) = 0 \quad \Rightarrow \cos x = \frac{1}{2} \ \Rightarrow\ x = \frac{\pi}{3} \ \text{ or } \frac{5\pi}{3}$$

The value of f at these critical points are

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3}$$

$$= \frac{5\pi}{3} + \sqrt{3}$$

$$\approx 6.968039$$

The values of f at the end points are $f(0) = 0$ and $f(2\pi) = 2\pi \approx 6.28$

Comparing these four numbers, the absolute minimum is $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$ and the absolute maximum is $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$. In this example both absolute minimum and absolute maximum occurs at the critical numbers.

Let us now see how the second derivatives of functions help determining the turning nature (of graphs of functions) and in optimization problems.

The second derivative test : Suppose f is continuous on an open interval that contains c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 5.49 : Discuss the curve $y = x^4 - 4x^3$ with respect to local extrema.

Solution : $f(x) = x^4 - 4x^3$

$$f'(x) = 4x^3 - 12x^2, \quad f''(x) = 12x^2 - 24x$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the second derivative test we evaluate the sign of f'' at these critical numbers.

$f''(0) = 0$, $f''(3) = 36 > 0$. Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum value and the point $(3, -27)$ is a minimum point. Since $f''(0) = 0$ the second derivative test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the first derivative test tells us that f does not have a local extremum at 0.

We summarise the above discussion as follows :

Procedure for finding and distinguishing stationary points.

(i) Given $y = f(x)$ determine $\frac{dy}{dx}$ (i.e., $f'(x)$)

(ii) Let $\frac{dy}{dx} = 0$ and solve for the critical numbers x.

(iii) Substitute the values of x into the original function $y = f(x)$ to find the corresponding y-coordinate values. This establish the co-ordinates of the stationary points. To determine the nature of the stationary points,

(iv) Find $\frac{d^2y}{dx^2}$ and substitute into it the values of x found in (ii).

If the result is :

(a) positive – the point is a minimum one

(b) negative – the point is a maximum one

(c) zero – the point cannot be an extremum (minimum or maximum)

OR

(v) Determine the sign of the gradient (slope $f'(x)$ of the curve just before and just after the stationary points. If the sign change for the gradient of the curve is

(a) positive to negative – this point is a maximum one

(b) negative to positive – this point is a minimum one

Example 5.50 : Locate the extreme point on the curve $y = 3x^2 - 6x$ and determine its nature by examining the sign of the gradient on either side.

Solution : Following the above procedure

(i) Since $y = 3x^2 - 6x$, $\frac{dy}{dx} = 6x - 6$

(ii) At a stationary point, $\frac{dy}{dx} = 0$, hence $x = 1$

(iii) When $x = 1$, $y = 3(1)^2 - 6(1) = -3$. Hence the coordinates of the stationary point is $(1, -3)$.

If x is slightly less than 1, say 0.9, then $\frac{dy}{dx} = 6(0.9) - 6 = -0.6 < 0$.

If x is slightly greater than 1, say 1.1 then $\frac{dy}{dx} = 6(1.1) - 6 = 0.6 > 0$.

Since the gradient (slope of the curve) changes its sign from negative to positive $(1, -3)$ is a minimum point.

Example 5.51 :

Find the local minimum and maximum values of $f(x) = x^4 - 3x^3 + 3x^2 - x$

Solution :

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$f'(x) = 4x^3 - 9x^2 + 6x - 1$$

At a turning point, $f'(x) = 0$ gives $4x^3 - 9x^2 + 6x - 1 = 0$

$$(x - 1)^2 (4x - 1) = 0 \Rightarrow x = 1, \ 1, \frac{1}{4}$$

When $x = 1$, $f(1) = 0$ and when $x = \frac{1}{4}$, $f\left(\frac{1}{4}\right) = \frac{-27}{256}$

Hence the coordinates of the stationary points are $(1, 0)$ and $\left(\frac{1}{4}, \frac{-27}{256}\right)$

$f''(x) = 12x^2 - 18x + 6 = 6(2x^2 - 3x + 1) = 6(x - 1)(2x - 1)$

When $x = 1$, $f''(1) = 0$. Thus the second derivative test gives no information about the extremum nature of f at $x = 1$.

When $x = \frac{1}{4}$, $f''\left(\frac{1}{4}\right) = \frac{9}{4} > 0$, hence $\left(\frac{1}{4}, \frac{-27}{256}\right)$ is a minimum point.

Caution :

No function will attain local maximum / minimum at the end points of its domain.

EXERCISE 5.9

(1) Find the critical numbers and stationary points of each of the following functions.

(i) $f(x) = 2x - 3x^2$ (ii) $f(x) = x^3 - 3x + 1$

(iii) $f(x) = x^{4/5}(x - 4)^2$ (iv) $f(x) = \dfrac{x + 1}{x^2 + x + 1}$

(v) $f(\theta) = \sin^2 2\theta$ in $[0, \pi]$

(vi) $f(\theta) = \theta + \sin\theta$ in $[0, 2\pi]$

(2) Find the absolute maximum and absolute minimum values of f on the given interval :

(i) $f(x) = x^2 - 2x + 2,\quad [0,3]$

(ii) $f(x) = 1 - 2x - x^2,\quad [-4,1]$

(iii) $f(x) = x^3 - 12x + 1,\quad [-3,5]$

(iv) $f(x) = \sqrt{9 - x^2},\quad [-1,2]$

(v) $f(x) = \dfrac{x}{x + 1},\quad [1,2]$

(vi) $f(x) = \sin x + \cos x,\quad \left[0, \dfrac{\pi}{3}\right]$

(vii) $f(x) = x - 2\cos x,\quad [-\pi, \pi]$

(3) Find the local maximum and minimum values of the following :

(i) $x^3 - x$ (ii) $2x^3 + 5x^2 - 4x$ (iii) $x^4 - 6x^2$

(iv) $(x^2 - 1)^3$ (v) $\sin^2\theta$, $[0, \pi]$ (vi) $t + \cos t$

5.10 Practical problems involving maximum and minimum values :

The methods we have learnt in this section for finding extreme values have practical applications in many areas of life. A business person wants to minimise costs and maximise profits. We also solve such problems as maximising areas, volumes and profits and minimising distances, times and costs. In solving such practical problems, the greatest challenge is often to convert the word problem into maximum – minimum problem by setting up the function that is to be maximised or minimised.

As a problem solving technique we suggest the following principles.

(1) **Understand the problem :** The first step is to read the problem carefully until it is clearly understood. Ask yourself what is the unknown? What are the given quantities? What are the given conditions?

(2) **Draw diagram :** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.

(3) **Introduce notation :** Assign a symbol to the quantity to be maximised or minimised, say Q. Also select symbols (a,b,c …,x, y, z) for the other unknown quantities and lable the diagram with these symbols.

(4) Express Q in terms of some other symbols from step 3.

(5) If Q has been expressed as a function of more than one variable in step 4, use the given information to find relationship (in the form of equation) among these variables. Then use these equations to eliminate all but one of these variables in the expression for Q.Thus Q will be given as a function of one variable x, say, $Q = f(x)$. Write the domain of this function.

(6) Use the methods discussed to find the absolute maximum or minimum value of f.

Remarks :

(1) If the domain is a closed interval then we apply the absolute max/min property to maximize / minimize the given function (see 5.52, 5.58).

(2) If the domain is an open interval then we apply either first derivative test (5.53) or second test for finding local max / min. Instead of first derivative one can also apply second derivative test if the second test exist. Similarly instead of second derivative test one can also apply first derivative test.

(3) All these cases ultimately lead us to the absolute max / min only.

Example 5.52 : A farmer has 2400 feet of fencing and want to fence of a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area ?

Solution :

We wish to maximize the area A of the rectangle. Let x and y be the width and length of the rectangle (in feet). Then we express A in terms of x and y as $A = xy$

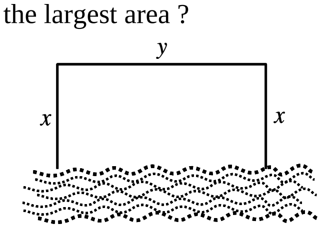


Fig. 5.38

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Therefore $2x + y = 2400$

Hence $y = 2400 - 2x$ and the area is $A = x(2400 - 2x) = 2400x - 2x^2$

Note that $x \geq 0$ and $x \leq 1200$ (otherwise $A < 0$). So the function that we wish to maximize is

$A(x) = 2400x - 2x^2, \quad 0 \leq x \leq 1200.$

$A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation $2400 - 4x = 0$ which gives x = 600. The maximum of A must occur either at this critical number or at an end point of the interval.

Since $A(0) = 0$, $A(600) = 7{,}20{,}000$ and $A(1200) = 0$, thus the maximum value is A (600) = 720,000. When $x = 600$, $y = 2400 - 1200 = 1200$

Hence the rectangular field should be 600 ft wide and 1200 ft long.

Note : This problem also be done by using second derivative test (local). In this case $x > 0$ and $y > 0$.

Example 5.53 :

Find a point on the parabola $y^2 = 2x$ that is closest to the point (1,4)

Solution : Let (x,y) be the point on the parabola $y^2 = 2x$. The distance between the points (1,4) and

(x,y) is $d = \sqrt{(x-1)^2 + (y-4)^2}$.

(x,y) lies on $y^2 = 2x \Rightarrow x = \dfrac{y^2}{2}$,

so $d^2 = f(y) = \left(\dfrac{y^2}{2} - 1\right)^2 + (y-4)^2$

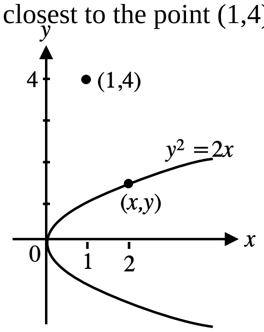


Fig. 5.39

(Note that the minimum of d occurs at the same point as the minimum of d^2)

$$\text{Now } f'(y) = 2\left(\frac{y^2}{2} - 1\right)(y) + 2(y-4)$$

$$= y^3 - 8 = 0 \text{ at a critical point.}$$

$$y^3 - 8 = 0 \Rightarrow y = 2 \text{ (since } y^2 + 2y + 4 = 0 \text{ is not possible)}$$

Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the first derivate test, for absolute extrema, the absolute minimum occurs when $y = 2$. The corresponding value of x is $x = \frac{y^2}{2} = 2$. Thus the point on $y^2 = 2x$ closest to (1,4) is (2,2).

Note : This problem also be done by using second derivative test

Example 5.54 :

Find the area of the largest rectangle that can be inscribed in a semi circle of radius r.

Solution :

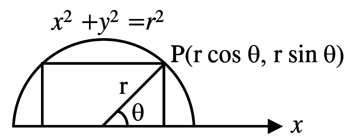


Fig. 5.40

Let θ be the angle made by OP with the positive direction of x–axis.

Then the area of the rectangle A is

$A(\theta) = (2\,r\cos\theta)\,(r\sin\theta)$

$= r^2\, 2\sin\theta\,\cos\theta = r^2 \sin 2\theta$

Now A(θ) is maximum when sin 2θ is maximum. The maximum value of $\sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{4}$. (Note that $A'(\theta) = 0$ when $\theta = \frac{\pi}{4}$)

Therefore the critical number is $\frac{\pi}{4}$. The area $A\left(\frac{\pi}{4}\right) = r^2$.

Note : The dimensions of the largest rectangle that can be inscribed in a semicircle are $\sqrt{2}r$, $\frac{r}{\sqrt{2}}$

Aliter : $A'(\theta) = 2r^2\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$; $\theta = \frac{\pi}{4}$

$A''(\theta) = -4r^2 \sin 2\theta < 0$, for $\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$ gives the maximum point and the maximum point is $\left(\frac{\pi}{4}, r^2\right)$

From the above problem, we understand that the method of calculus gives the solution faster than the algebraic method.

Example 5.55 : The top and bottom margins of a poster are each 6 cms and the side margins are each 4 cms. If the area of the printed material on the poster is fixed at 384 cms^2, find the dimension of the poster with the smallest area.

Solution : Let x and y be the length and breadth of printed area, then the area $xy = 384$

Dimensions of the poster area are $(x + 8)$ and $(y + 12)$ respectively.

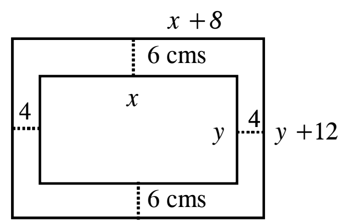


Fig. 5.41

Poster area $A = (x + 8)(y + 12)$

$$= xy + 12x + 8y + 96$$

$$= 12x + 8y + 480$$

$$= 12x + 8\left(\frac{384}{x}\right) + 480$$

$$A' = 12 - 8 \times 384 \times \frac{1}{x^2}$$

$$A'' = 16 \times 384 \times \frac{1}{x^3}$$

$$A' = 0 \Rightarrow x = \pm 16$$

But $x > 0$

$$\therefore x = 16$$

when $x = 16$, $A'' > 0$

$\therefore$ when $x = 16$, the area is minimum

$$\therefore y = 24$$

$\therefore x + 8 = 24, y + 12 = 36$

Hence the dimensions are 24cm and 36 cm.

Example 5.56 : Show that the volume of the largest right circular cone that can be inscribed in a sphere of radius a is $\frac{8}{27}$ (volume of the sphere).

Solution : Given that a is the radius of the sphere and let x be the base radius of the cone. If h is the height of the cone, then its volume is

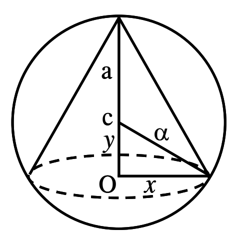


Fig. 5.42

$$V = \frac{1}{3}\pi x^2 h$$

$$= \frac{1}{3}\pi x^2 (a + y) \qquad \ldots(1)$$

where $OC = y$ so that height $h = a + y$.

From the diagram $x^2 + y^2 = a^2$ (2)

Using (2) in (1) we have

$$V = \frac{1}{3}\pi (a^2 - y^2)(a + y)$$

For the volume to be maximum :

$$V' = 0 \Rightarrow \frac{1}{3}\pi\,[a^2 - 2ay - 3y^2] = 0$$

$$\Rightarrow 3y = +a \text{ or } y = -a$$

$$\Rightarrow y = \frac{a}{3} \text{ and } y = -a \text{ is not possible}$$

$$\text{Now } V'' = -\pi\frac{2}{3}(a + 3y) < 0 \text{ at } y = \frac{a}{3}$$

$\therefore$ the volume is maximum when y = $\frac{a}{3}$ and the maximum volume is

$$\frac{1}{3}\pi \times \frac{8a^2}{9}\left(a + \frac{1}{3}a\right) = \frac{8}{27}\left(\frac{4}{3}\pi a^3\right) = \frac{8}{27} \text{ (volume of the sphere)}$$

Example 5.57 : A closed (cuboid) box with a square base is to have a volume of 2000 c.c. The material for the top and bottom of the box is to cost Rs. 3 per square cm. and the material for the sides is to cost Rs. 1.50 per square cm. If the cost of the materials is to be the least, find the dimensions of the box.

Solution : Let x, y respectively denote the length of the side of the square base and depth of the box. Let C be the cost of the material

Area of the bottom = x^2

Area of the top = x^2

Combined area of the top and bottom = $2x^2$

Area of the four sides = $4xy$

Cost of the material for the top and bottom = $3(2x^2)$

Cost of the material for the sides = $(1.5)(4xy) = 6xy$

Total cost C = $6x^2 + 6xy$ …(1)

Volume of the box V = (area) (depth) = $x^2y = 2000$ …(2)

Eliminating y from (1) & (2) we get $C(x) = 6x^2 + \frac{12000}{x}$ …(3)

where $x > 0$, ie., $x \in (0, +\infty)$ and $C(x)$ is continuous on $(0, +\infty)$.

$$C'(x) = 12x - \frac{12000}{x^2}$$

$$C'(x) = 0 \Rightarrow 12x^3 - 12000 = 0 \Rightarrow 12(x^3 - 10^3) = 0$$

$$\Rightarrow x = 10 \text{ or } x^2 + 10x + 100 = 0$$

$x^2 + 10x + 100 = 0$ is not possible

$\therefore$ The critical numbers is $x = 10$.

Now $C''(x) = 12 + \dfrac{24000}{x^3}$; $C''(10) = 12 + \dfrac{24000}{1000} = 36 > 0$

$\therefore$ C is minimum at $(10, C(10)) = (10, 1800)$ $\therefore$ the base length is 10cm and depth is $y = \dfrac{2000}{100} = 20$ cm.

Example 5.58 :

A man is at a point P on a bank of a straight river, 3 km wide, and wants to reach point Q, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point R and then run to Q, or he could row directly to Q, or he could row to some point S between Q and R and then run to Q. If he can row at 6 km/h and run at 8 km/h where should he land to reach Q as soon as possible ?

Solution :

Let x be the distance from R to S. Then the running distance is $8 - x$ and the distance $PS = \sqrt{x^2 + 9}$. We know that time $= \dfrac{\text{distance}}{\text{rate}}$.

Then the rowing time

$R_t = \dfrac{\sqrt{x^2+9}}{6}$ and the running time $r_t = \dfrac{(8-x)}{8}$

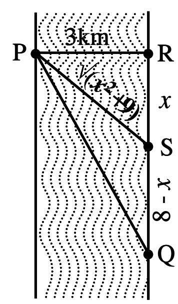


Fig. 5.43

Therefore the total time $T = R_t + r_t = \dfrac{\sqrt{x^2+9}}{6} + \dfrac{(8-x)}{8}$, $0 \le x \le 8$.

Notice that if $x = 0$, he rows to R and if $x = 8$ he rows directly to Q.

$$T'(x) = 0 \Rightarrow \quad T'(x) = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8} = 0 \text{ for critical points.}$$

$$4x = 3\sqrt{x^2+9}$$

$$16x^2 = 9(x^2+9)$$

$$7x^2 = 81$$

$$\Rightarrow \; x = \frac{9}{\sqrt{7}} \quad \text{since } x = -\frac{9}{\sqrt{7}} \text{ is not admissible.}$$

The only critical number is $x = \frac{9}{\sqrt{7}}$. We calculate T at the end point of the domain 0 and 8 and at $x = \frac{9}{\sqrt{7}}$.

$$T(0) = 1.5, \quad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33, \text{ and } T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T occurs when $x = \frac{9}{\sqrt{7}}$, the man should land the boat at a point $\frac{9}{\sqrt{7}}$ km (≈ 3.4 km) down stream from his starting point.

EXERCISE 5.10

(1) Find two numbers whose sum is 100 and whose product is a maximum.

(2) Find two positive numbers whose product is 100 and whose sum is minimum.

(3) Show that of all the rectangles with a given area the one with smallest perimeter is a square.

(4) Show that of all the rectangles with a given perimeter the one with the greatest area is a square.

(5) Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

(6) Resistance to motion, F, of a moving vehicle is given by, $F = \frac{5}{x} + 100x$.

Determine the minimum value of resistance.

5.11 Concavity (convexity) and points of inflection :

Figure 5.44 (a), (b) shows the graphs of two increasing functions on $[a, b]$. Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behaviour? In *fig. 5.44 (c), (d)* tangents to these curves have been drawn at several points. In (a) the curve lies above the tangents and f is called concave upward (convex downward) on $[a, b]$. In (b) the curve lies below the tangents and g is called concave downward (convex upward) on $[a, b]$

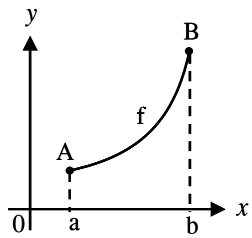


Fig. 5.44 (a)

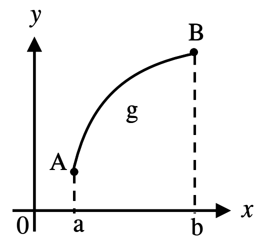


Fig. 5.44 (b)

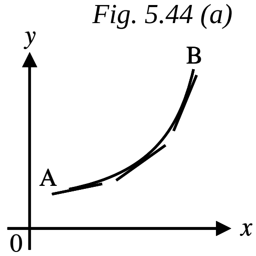


Fig. 5.44(c)

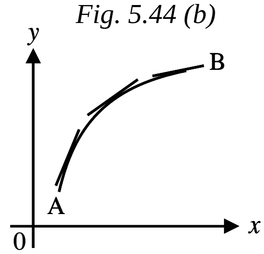


Fig. 5.44 (d)

Definition :

If the graph of f lies above all of its tangents on an interval I, then it is called concave upward (convex downward) on I. If the graph of f lies below all of its tangents on I, it is called concave downward (convex upward) on I.

Let us see how the second derivative helps to determine the intervals of concavity (convexity). Looking at *Fig.5.44(c)*, you can see that, going from left to right, the slope of the tangent increases. This means that the derivative $f'(x)$ is an increasing function and therefore its derivative $f''(x)$ is positive. Likewise in *Fig.5.44 (d)* the slope of the tangent decreases from left to right, so $f'(x)$ decreases and therefore $f''(x)$ is negative. This reasoning can be reversed and suggests that the following theorem is true.

The test for concavity (convexity) :

Suppose f is twice differentiable on an interval I.

(i) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward (convex downward) on I.

(ii) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward (convex upward) on I.

Definition : A point P on a curve is called a point of inflection if the curve changes from concave upward (convex downward) to concave downward (convex upward) or from concave downward (convex upward) to concave upward (convex downward) at P.

That is the point that separates the convex part of a continuous curve from the concave part is called the point of inflection of the curve.

It is obvious that at the point of inflection the tangent line, if it exists, cuts the curve, because on one side the curve lies under the tangent and on the other side, above it. The following theorem says under what situation a critical point of f' becomes a point of inflection.

Theorem :

Let a curve be defined by an equation $y = f(x)$. If $f''(x_0) = 0$ or $f''(x_0)$ does not exist and if the derivative $f''(x)$ changes sign when passing through $x = x_0$, then the point of the curve with abcissa $x = x_0$ is the point of inflection. Equivalently the point $(x_0, f(x_0))$ is a point of inflection of the graph of f if there exists a neighbourhood (a, b) of x_0 such that $f''(x) > 0$ for every x in (a, x_0) and $f''(x) < 0$ for every x in (x_0, b) or vice versa. That is in the neighbourhood of x_0, $f''(a)$ and $f''(b)$ differ in sign.

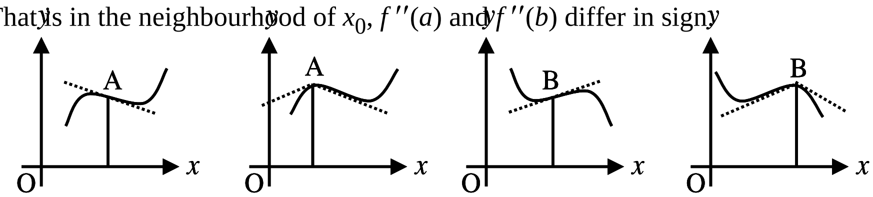


Fig. 5.45

Remark :

We caution the reader that points of inflections need not be critical points and critical points need not be points of inflections. However $x = x_0$ is a critical point such that $f'(x)$ does not change its sign as $f(x)$ passes through x_0, then x_0 is a point of inflection and for points of inflections x_0, it is necessary that $f''(x_0) = 0$. If $f''(x)$ does not change its sign even if $f''(x_0) = 0$ then x_0 cannot be a point of inflection. Thus the conjoint of the above discussion is that for points of inflections x_0, $f''(x_0) = 0$ and in the immediate neighbourhood (a, b) of x_0, $f''(a)$ and $f''(b)$ must differ in sign.

If $x = x_0$ is a root of odd order – simple, triple, etc. of the function $f'(x) = 0$, then $x = x_0$ yields a maximum or minimum. If $x = x_0$ is a root of even order, $x = x_0$ yield a point of inflection with a horizontal tangent. These concepts are made clear in the following illustrative example $y = x^3$.

$y' = 3x^2$ and $y'' = 6x$.

Here $y'(0) = 0$ and $y''(0) = 0$ and $x = 0$ happens to be a critical point of both y and y'. Clearly $y'(x) > 0$ for $x < 0$ and $x > 0$. Thus y' does not change its sign as $f(x)$ passes through $x = 0$.

That is $y'(-0.1) > 0$ and $y'(0.1) > 0$ i.e., in the neighbourhood $(-0.1, 0.1)$ of 0, y' does not change its sign. Thus the first derivative test confirms that (0, 0) is a point of inflection.

Again $y''(0) = 0$, $y''(-0.1) < 0$ and $y''(0.1) > 0$. Here y'' changes its sign as $y(x)$ passes through $x = 0$. In this case the second derivative (concavity) test also confirms that (0, 0) is a point of inflection. Note that (0, 0) separates the convex part of $y = x^3$ from the concave part.

Note also that $y'(x) = 3x^2$ and $x = 0$ is a double root of $y'(x) = 0$. The root order test also confirms that (0, 0) is a point of inflection with x-axis as the horizontal tangent at (0, 0)

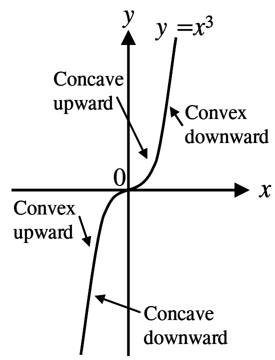


Fig. 5.46

Example 5.59 :

Determine the domain of concavity (convexity) of the curve $y = 2 - x^2$.

Solution : $y = 2 - x^2$

$y' = -2x$ and $y'' = -2 < 0$ for $x \in R$

Here the curve is everywhere concave downwards (convex upwards).

Example 5.60 :

Determine the domain of convexity of the function $y = e^x$.

Solution : $y = e^x$; $y'' = e^x > 0$ for x

Hence the curve is everywhere convex downward.

Example 5.61 : Test the curve $y = x^4$ for points of inflection.

Solution :

$$y = x^4$$

$$y'' = 12x^2 = 0 \text{ for } x = 0$$

and $y'' > 0$ for $x < 0$ and $x > 0$

Therefore the curve is concave upward and y'' does not change sign as $y(x)$ passes through $x = 0$. Thus the curve does not admit any point of inflection.

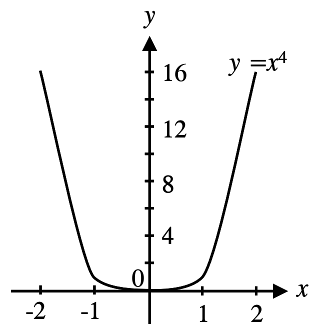


Fig. 5.47

Note : The curve is concave upward in $(-\infty, 0)$ and $(0, \infty)$.

Example 5.62 : Determine where the curve $y = x^3 - 3x + 1$ is cancave upward, and where it is concave downward. Also find the inflection points.

Solution :

$$f(x) = x^3 - 3x + 1$$

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

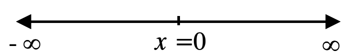


Fig. 5.48

Now $f''(x) = 6x$

Thus $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$.

The test for concavity then tells us that the curve is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$. Since the curve changes from concave downward to concave upward when $x = 0$, the point $(0, f(0))$ i.e., $(0, 1)$ is a point of inflection. Note that $f''(0) = 0$

Example 5.63 :

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity and points of inflection.

Solution :

$$f(x) = x^4 - 4x^3 \Rightarrow f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into three intervals.

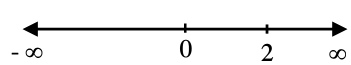


Fig. 5.49

$(-\infty, 0)$, $(0, 2)$, $(2, \infty)$ and complete the following chart.

Inerval	$f''(x) = 12x\,(x-2)$	concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	–	downward
$(2, \infty)$	+	upward

The point $(0, f(0))$ i.e., $(0, 0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also $(2, f(2))$ i.e., $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Note : The intervals of concavity can be obtained by taking and checking a sample point in the sub-interval.

Example 5.64 : Find the points of inflection and determine the intervals of convexity and concavity of the Gaussion curve $y = e^{-x^2}$

Solution : $y' = -2xe^{-x^2}$; $y'' = 2e^{-x^2}(2x^2-1)$

(The first and second derivatives exist everywhere). Find the values of x for which $y'' = 0$

$$2e^{-x^2}(2x^2-1) = 0$$

$$x = -\frac{1}{\sqrt{2}}, \quad \text{or} \quad x = \frac{1}{\sqrt{2}}$$

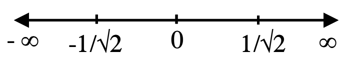


Fig. 5.50

when $x < -\frac{1}{\sqrt{2}}$ we have $y'' > 0$ and when $x > -\frac{1}{\sqrt{2}}$ we have $y'' < 0$

The second derivative changes sign from positive to negative when passing through the point $x = -\frac{1}{\sqrt{2}}$. Hence, for $x = -\frac{1}{\sqrt{2}}$, there is a point of inflection on the curve; its co-ordinates are $\left(-\frac{1}{\sqrt{2}},\ e^{-\frac{1}{2}}\right)$

When $x < \frac{1}{\sqrt{2}}$ we have $y'' < 0$ and when $x > \frac{1}{\sqrt{2}}$ we have $y'' > 0$. Thus there is also a point of inflection on the curve for $x = \frac{1}{\sqrt{2}}$; its co-ordinates are $\left(\frac{1}{\sqrt{2}},\ e^{-\frac{1}{2}}\right)$. (Incidentally, the existence of the second point of inflection follows directly from the symmetry of the curve about the y-axis). Also from the signs of the second derivatives, it follows that for $-\infty < x < -\frac{1}{\sqrt{2}}$ the curve is concave upward ;

for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ the curve is convex upward ;

for $\frac{1}{\sqrt{2}} < x < \infty$ the curve is concave upward.

Example 5.65 :

Determine the points of inflection if any, of the function $y = x^3 - 3x + 2$

Solution :

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x+1)(x-1)$$

$$\frac{d^2y}{dx^2} = 6x = 0 \Rightarrow x = 0$$

$$\text{Now } \frac{d^2y}{dx^2}(-0.1) = 6(-0.1) < 0 \text{ and}$$

$$\frac{d^2y}{dx^2}(0.1) = 6(0.1) > 0.$$ In the neighbourhood $(-0.1, 0.1)$ of 0, $y''(-0.1)$ and $y''(0.1)$ are of opposite signs. Therefore $(0, y(0))$ i.e., $(0, 2)$ is a point of inflection.

Note : Note that $x = 0$ is not a critical point since $y'(0) = -3 \neq 0$.

Example 5.66 :

Test for points of inflection of the curve $y = \sin x$, $x \in (0, 2\pi)$

Solution :

$$y' = \cos x$$

$$y'' = -\sin x = 0 \Rightarrow x = n\pi,\ n = 0, \pm 1, \pm 2, \ldots$$

since $x \in (0, 2\pi)$, $x = \pi$ corresponding to $n = 1$.

$$\text{Now } y''(.9\pi) = -\sin(.9\pi) < 0 \text{ and}$$

$$y''(1.1\pi) = -\sin(1.1\pi) > 0 \text{ since } \sin(1.1\pi) \text{ is negative}$$

The second derivative test confirms that $(\pi, f(\pi)) = (\pi, 0)$ is a point of inflection.

Note : Note that $x = \pi$ is not a stationary point since $y'(\pi) = \cos \pi = -1 \neq 0$. In fact $y = \sin x$ admits countable number of points of inflections in the range $(-\infty, \infty)$, each of which is given by $(n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$ and in none of the cases, $y'(n\pi) = (-1)^n$ vanishes. This shows that points of inflections need not be stationary points.

EXERCISE 5.11

Find the intervals of concavity and the points of inflection of the following functions :

(1) $f(x) = (x-1)^{1/3}$

(2) $f(x) = x^2 - x$

(3) $f(x) = 2x^3 + 5x^2 - 4x$

(4) $f(x) = x^4 - 6x^2$

(5) $f(\theta) = \sin 2\theta$ in $(0, \pi)$

(6) $y = 12x^2 - 2x^3 - x^4$

Testing a differentiable function for maximum and minimum with a first derivative

This gives us the following diagram of possible cases.

Signs of derivative $f'(x)$ when passing through critical point x_0			Character of critical point
$x < x_0$	$x = x_0$	$x > x_0$	
+	$f'(x_0) = 0$ or is discontinuous	–	Maximum point
–	$f'(x_0) = 0$ or is discontinuous	+	Minimum point
+	$f'(x_0) = 0$ or is discontinuous	+	Neither maximum nor minimum (function increases). But is a point of inflection.
–	$f'(x_0) = 0$ or is discontinuous	–	Neither maximum nor minimum (function decreases) But is a point of inflection.

Second derivative test

This gives us the following diagram of possible cases.

Signs of derivative $f''(x)$ at the critical point of $f(x)$ or $f'(x)$				Character of the point
	$x = x_0$			
	$f'(x_0)$	$f''(x_0)$		
	0	–	Critical point of f	Maximum point
	0	+	Critical point of f	Minimum point
$x < x_0$		$f''(x_0)$	$x > x_0$	
+	0 or $\neq 0$	0	–	Point of Inflection
–	0 or $\neq 0$	0	+	Point of inflection
+	0 or $\neq 0$	0	+	Unknown
–	0 or $\neq 0$	0	–	Unknown

6. DIFFERENTIAL CALCULUS APPLICATIONS-II

6.1 Differentials : Errors and Approximation

We have used the Liebnitz notation $\frac{dy}{dx}$ to denote the derivative of y with respect to x but we have regarded it as a single entity and not as a ratio. In this section we give the quantities dy and dx separate meanings in such a way that their ratio is equal to the derivative. We also see that these quantities, called differentials, are useful in finding the approximate values of functions.

Definition 1 : Let $y = f(x)$ be a differentiable function. Then the quantities dx and dy are called differentials. The differential dx is an independent variable that is dx can be given any real number as the value. The differential dy is then defined in terms of dx by the relation

$$dy = f'(x)\, dx \quad (dx \approx \Delta x)$$

Note :

(1) The differentials dx and dy are both variables, but dx is an independent variable, where as dy is a dependent variable – it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

(2) If $dx \neq 0$ we can divide both sides of $dy = f'(x)\, dx$ by dx to obtain $\frac{dy}{dx} = f'(x)$. Thus $\frac{dy}{dx}$ now is the ratio of differentials.

Example 6.1 : If $y = x^3 + 2x^2$ (i)find dy

(ii) find the value of dy when $x = 2$ and $dx = 0.1$

Solution :

(i) If $f(x) = x^3 + 2x^2$, then $f'(x) = 3x^2 + 4x$, so $dy = (3x^2 + 4x)\, dx$

(ii) Substituting $x = 2$ and $dx = 0.1$, we get $dy = (3 \times 2^2 + 4 \times 2)0.1 = 2.$

6.1.1 Geometric meaning of differentials :

Let P(x,$f(x)$) and Q($x + \Delta x$, $f(x + \Delta x)$) be points on the graph of f and set $dx = \Delta x$. The corresponding change in y is $\Delta y = f(x + \Delta x) - f(x)$

The slope of the tangent line PR is the derivate $f\,'(x)$. Thus the directed distance from S to R is $f\,'(x)\, dx = dy$.

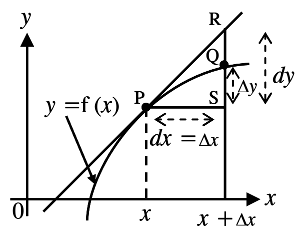


Fig. 6.1

Therefore dy represents the amount that the tangent line rises or falls whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Since $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$, we have $\dfrac{\Delta y}{\Delta x} \approx \dfrac{dy}{dx}$ ….(1) when Δx is small.

Geometrially, this says that the slope of the secant line PQ is very close to the slope of the tangent line at P when Δx is small. If we take $dx = \Delta x$, then (1) becomes $\Delta y \approx dy$ ….(2) which says that if Δx is small, then the actual change in y is approximately equal to the differential dy. Again this is geometrically evident in the case illustrated by *Fig. 6.1*. The actual change in y is referred as **absolute error**.

The actual error in y is $\Delta y \approx dy$.

The quantity $\dfrac{\Delta y}{y} = \dfrac{\text{Actual change in } y}{\text{Actual value of } y}$ is called **relative error** and the quantity $\left(\dfrac{\Delta y}{y}\right) \times 100$ is called **percentage error**.

The approximation given by (2) can be used in computing approximate values of functions. Suppose that $f(a)$ is a known number and an approximate value is calculated for $f(a + \Delta x)$ where dx is small, since $f(a + \Delta x) = f(a) + \Delta y$, (2) gives, $f(a + \Delta x) \approx f(a) + dy$….(3)

Example 6.2 : Compute the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ where x changes *(i)* from 2 to 2.05 and *(ii)* from 2 to 2.01

Solution *:*

(i) We have $f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$

$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625.$

and $\Delta y = f(2.05) - f(2) = 0.717625.$

In general $dy = f\,'(x)\, dx = (3x^2 + 2x - 2)dx$

When $x = 2$, $dx = \Delta x = 0.05$ and $dy = [(3(2)^2+2(2)-2]\, 0.05 = 0.7$

(ii) $f(2.01) = (2.01)^3 - (2.01)^2 - 2(2.01) + 1 = 9.140701$

$\therefore\ \Delta y = f(2.01) - f(2) = 0.140701$

When $dx = \Delta x = 0.01,\ dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$

Remark : The approximation $\Delta y \approx dy$ becomes better as Δx becomes smaller in Example 6.2. Also dy was easier than to compute Δy. For more complicated functions it may be impossible to compute Δy exactly. In such cases the approximation by differentials is especially useful.

Example 6.3 : Use differentials to find an approximate value for $\sqrt[3]{65}$.

Solution : Let $y = f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Then $dy = \frac{1}{3}x^{\frac{-2}{3}}\,dx$

Since $f(64) = 4$. We take $x = 64$ and $dx = \Delta x = 1$

This gives $dy = \frac{1}{3}\,(64)^{\frac{-2}{3}}\,(1) = \frac{1}{3(16)} = \frac{1}{48}$

$$\therefore\ \sqrt[3]{65} = f(64 + 1) \approx f(64) + dy = 4 + \frac{1}{48} \approx 4.021$$

Note : The actual value of $\sqrt[3]{65}$ is 4.0207257... Thus the approximation by differentials is accurate to three decimal places even when $\Delta x = 1$.

Example 6.4 : The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of atmost 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere ?

Solution : If the radius of the sphere is r, then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then, the corresponding error in the calculated value of V is ΔV. which can be approximated by the differential $dV = 4\pi r^2\,dr$.

When $r = 21$ and dr=0.05, this becomes $dV = 4\pi(21)^2\,0.05 \approx 277$.

The maximum error in the calculated volume is about 277 cm^3.

Note : Although the possible error in the above example may appear to be rather large, a better picture of the error is given by the relative error, which is computed by dividing the error by the total volume.

$$\frac{\Delta V}{V} \approx \frac{dV}{V} \approx \frac{277}{38{,}808} \approx 0.00714$$

Thus a relative error of $\frac{dr}{r} = \frac{0.05}{21} \approx 0.0024$ in the radius produces a relative error of about 0.007 in the volume. The errors could also be expressed as percentage errors of 0.24% in the radius and 0.7% in the volume.

Example 6.5 : The time of swing T of a pendulum is given by $T = k\sqrt{l}$ where k is a constant. Determine the percentage error in the time of swing if the length of the pendulum l changes from 32.1 cm to32.0 cm.

Solution : If $T = k\sqrt{l} = k\, l^{\frac{1}{2}}$

$$\text{Then } \frac{dT}{dl} = k\left(\frac{1}{2} \times l^{-\frac{1}{2}}\right) = \left(\frac{k}{2\sqrt{l}}\right) \text{ and } dl = 32.0 - 32.1 = -0.1 \text{ cm}$$

Error in T = Approximate change in T.

$$\Delta T \approx dT = \left(\frac{dT}{dl}\right) dl = \left(\frac{k}{2\sqrt{l}}\right) (-0.1)$$

$$\text{Percentage error} = \left(\frac{\Delta T}{T}\right) \times 100\ \% = \frac{\frac{k}{2\sqrt{l}}(-0.1)}{k\sqrt{l}} \times 100\ \%$$

$$= \left(\frac{-0.1}{2l}\right) \times 100\ \% = \left(\frac{-0.1}{2(32.1)}\right) \times 100\%$$

$$= -0.156\%$$

Hence the percentage error in the time of swing is a decrease of 0.156%.

Aliter : $T = k\sqrt{l}$

Taking log on both sides, $\log T = \log k + \frac{1}{2} \log l$

Taking differential on both sides, $\frac{1}{T}\, dT = 0 + \frac{1}{2}\ \frac{1}{l} \times dl$

$$\text{i.e, } \frac{\Delta T}{T} \approx \frac{1}{T}\, dT = 0 + \frac{1}{2}\ \frac{1}{l} \times dl$$

$$\frac{\Delta T}{T} \times 100 = \frac{1}{2} \times \frac{dl}{l} \times 100$$

$$= \frac{1}{2} \times \frac{(-0.1)}{32.1} \times 100$$

$$= -0.156\%$$

ie., the percentage error in the time of swing is a decrease of 0.156.

Caution : Differentiation is carried out with the common understanding that the function involved admit logarithmic differentiation.

Example 6.6 : A circular template has a radius of 10 cm (± 0.02). Determine the possible error in calculating the area of the templates. Find also the percentage error.

Solution : Area of circular template $A = \pi r^2$, hence $\frac{dA}{dr} = 2\pi r$, Approximate change in area $\Delta A \approx (2\pi r)dr$. When $r = 10$ *cm* and $dr = 0.02$

$\Delta A = (2\pi\ 10)\ (0.02) \approx 0.4\pi\ cm^2$ i.e, the possible error in calculating the template area is approximately 1.257 cm^2

$$\text{Percentage error} \approx \left(\frac{0.4\pi}{\pi(10)^2}\right) \times 100 = 0.4\%$$

Example 6.7 : Show that the percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the percentage error in the number .

Solution : Let x be the number. Let $y = f(x) = (x)^{\frac{1}{n}}$

$$\text{Then } \log y = \frac{1}{n} \log x$$

Taking differential on both sides, we have $\frac{1}{y} dy = \frac{1}{n} \times \frac{1}{x}\ dx$

$$i.e., \quad \frac{\Delta y}{y} \approx \frac{1}{y} dy = \frac{1}{n} \cdot \frac{1}{x}\ dx$$

$$\therefore \frac{\Delta y}{y} \times 100 \approx \frac{1}{n}\left(\frac{dx}{x} \times 100\right)$$

$$= \frac{1}{n} \text{ times the percentage error in the number.}$$

Example 6.8 : Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 1%

Solution : The volume of the cube of side x is,

$$V = x^3 \quad ; \quad dV = 3x^2\ dx$$

$$\text{When } dx = 0.01x, \quad dV = 3x^2 \times (0.01x) = 0.03\ x^3\ m^3.$$

EXERCISE 6.1

(1) Find the differential of the functions

(i) $y = x^5$ (ii) $y = \sqrt[4]{x}$ (iii) $y = \sqrt{x^4 + x^2 + 1}$

(iv) $y = \frac{x-2}{2x+3}$ (v) $y = \sin 2x$ (vi) $y = x \tan x$

(2) Find the differential dy and evaluate dy for the given values of x and dx.

(i) $y = 1 - x^2,\ x = 5,\ dx = \frac{1}{2}$

(ii) $y = x^4 - 3x^3 + x - 1,\ x = 2,\ dx = 0.1.$

(iii) $y = (x^2 + 5)^3,\ \ x = 1,\ \ dx = 0.05$

(iv) $y = \sqrt{1 - x},\ x = 0,\ dx = 0.02$

(v) $y = \cos x,\ x = \frac{\pi}{6}\ \ dx = 0.05$

(3) Use differentials to find an approximate value for the given number

(i) $\sqrt{36.1}$ (ii) $\frac{1}{10.1}$

(iii) $y = \sqrt[3]{1.02} + \sqrt[4]{1.02}$ (iv) $(1.97)^6$

(4) The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error in computing (i) the volume of the cube and (ii) the surface area of cube.

(5) The radius of a circular disc is given as 24 cm with a maximum error in measurement of 0.02 cm.

(i) Use differentials to estimate the maximum error in the calculated area of the disc.

(ii) Compute the relative error ?

6.2 Curve Tracing :

The study of calculus and its applications is best understood when it is studied through the geometrical representation of the functions involved. In order to investigate the nature of a function (graph) it is not possible to locate each and every point of the graph. But we can sketch the graph of the function and know its nature by certain specific properties and some special points. To do this we adopt the following strategies.

(1) Domain, Extent, Intercepts and origin :

(i) Domain of a function $y = f(x)$ is determined by the values of x for which the function is defined.

(ii) Horizontal (vertical) extent of the curve is determined by the intervals of x (y) for which the curve exists.

(iii) $x = 0$ yields the y – intercept and $y = 0$ yields the x – intercept

(iv) If (0,0) satisfies the given equation then the curve will pass through the origin.

(2) Symmetry : Find out whether the curve is symmetrical about any line with the help of the following rules :

The curve is symmetrical about

(i) the x-axis if its equation is unaltered when y is replaced by $-y$

(ii) the y-axis if its equation is unaltered when x is replaced by $-x$.

(iii) the origin if it is unaltered when x is replaced by $-x$ and y is replaced by $-y$ simultaneously.

(iv) the line $y = x$ if its equation is unchanged when x and y are replaced by y and x.

(v) the line $y = -x$ if its equation is unchanged when x and y are replaced by $-y$ and $-x$.

(3) Asymptotes (parallel to the co-ordinate axes only) :

If $y \to c$, c finite [$x \to k$, k finite] whenever $x \to \pm\infty$ [$y \to \pm\infty$] then the line $y = c$ [$x = k$] is an asymptote parallel to x – axis [y – axis].

(4) Monotonicity : Determine the intervals of x for which the curve is decreasing or increasing using the first derivates test.

(5) Special points (Nature of bending) :

Determine the intervals of concavity and inflection points using the first and second derivatives test.

Illustrative Example :

Example 6.9 : Trace the curve $y = x^3 + 1$

Solution :

(1) Domain, Extent, intercepts and origin :

The function is defined for all real values of x and hence the domain is the entire interval $(-\infty, \infty)$. Horizontal extent is $-\infty < x < \infty$ and vertical extent is $-\infty < y < \infty$. Clearly $x = 0$ yields the y intercept as + 1 and y = 0 yields the x intercepts as -1. It is obvious that the curve does not pass through (0,0).

(2) **Symmetry Test :** The symmetry test shows that the curve does not possess any of the symmetry properties.

(3) **Asymptotes :** As $x \to c$ (for c finite) y does not tend to $\pm\infty$ and vice versa. Therefore the curve doest not admit any asymptote.

(4) **Monotonicity :** The first derivative test shows that the curve is increasing throughout $(-\infty,\infty)$ since $y' \geq 0$ for all x.

(5) **Special points :** The curve is concave downward in $(-\infty, 0)$ and concave upward in $(0, \infty)$ since

$y'' = 6x < 0$ for $x < 0$

$y'' = 6x > 0$ for $x > 0$ and

$y'' = 0$ for $x = 0$ yields $(0,1)$

as the inflection point

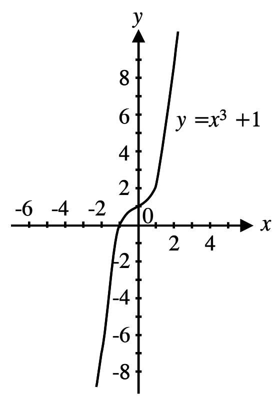


Fig. 6.2

Example 6.10 : Trace the cure $y^2 = 2x^3$.

Solution :

(1) **Domain, extent, Intercept and Origin :**

When $x \geq 0$, y is well defined. As $x \to \infty$, $y \to \pm\infty$,

The curve exists in first and fourth quadrant only

The intercepts with the axes are given by :

$x = 0$, $y = 0$ and when $y = 0$, $x = 0$

Clearly the curve passes through origin.

(2) **Symmetry :** By symmetry test, we have, the curve is symmetric about x – axis only.

(3) **Asymptotes :** As $x \to +\infty$, $y \to \pm\infty$, and vice versa.

$\therefore$ the curve does not admit asymptotes.

(4) **Monotonicity :** For the branch $y = \sqrt{2}\, x^{3/2}$ of the curve is increasing since $\frac{dy}{dx} > 0$ for $x > 0$ and the branch $y = -\sqrt{2}x^{3/2}$ of the curve is decreasing since $\frac{dy}{dx} < 0$ for $x > 0$

(5) Special points : (0,0) is not a point of inflection.

This curve is called a semi – cubical parabola.

Note :

(0, 0) admits a pair of tangents which coincide, resulting in a special point, called cusp.

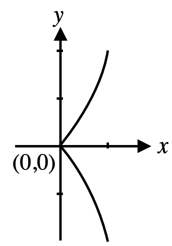


Fig. 6.3

Example 6.11 : Discuss the curve $y^2 (1 + x) = x^2 (1 - x)$

for (i) existence (ii) symmetry (iii) asymptotes (iv) loops

Solution :

(i) Existence : The function is not well defined when $x > 1$ and $x \le -1$ and the curve lies between $-1 < x \le 1$.

(ii) Symmetry : The curve is symmetrical about the x – axis only.

(iii) Asymptotes : $x = -1$ is a vertical asymptote to the curve parallel to y – axis.

(iv) Loops : (0,0) is a point through which the curve passes twice and hence a loop is formed between $x = 0$ and $x = 1$.

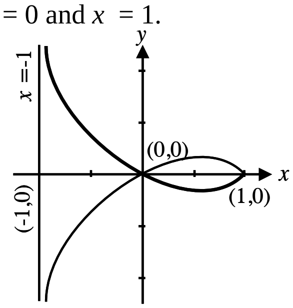


Fig. 6.4

Example 6.12 : Discuss the curve $a^2 y^2 = x^2 (a^2 - x^2)$, $a > 0$

for (i) existence (ii) symmetry (iii) asymptotes (iv) loops

Solution :

(i) Existence :

The curve is well defined for $(a^2 - x^2) \ge 0$ i.e., $x^2 \le a^2$ i.e., $x \le a$ and $x \ge - a$

(ii) Symmetry : The curve is symmetrical about x-axis, y – axis, and hence about the origin.

(iii) Asymptotes : It has no asymptote.

(iv) Loops : For $-a < x < 0$ and $0 < x < a$, $y^2 > 0 \Rightarrow y$ is positive and negative $\therefore$ a loop is formed between $x = 0$ and $x = a$ and another loop is formed between $x = -a$ and $x = 0$.

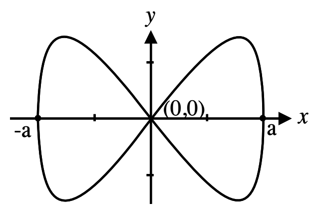


Fig. 6.5

Example 6.13 : Discuss the curve $y^2 = (x-1)(x-2)^2$.
for (i) existence (ii) symmetry (iii) asymptotes (iv) loops

Solution :

(i) Existence :

The curve is not defined for $x - 1 < 0$, ie., whenever $x < 1$, the R.H.S. is negative $\Rightarrow y^2 < 0$ which is impossible. The curve is defined for $x \geq 1$.

(ii) Symmetry : The curve is symmetrical about x-axis.

(iii) Asymptote : The curve does not admit asymptotes.

(iv) Loops : Clearly a loop is formed between (1, 0) and (2, 0).

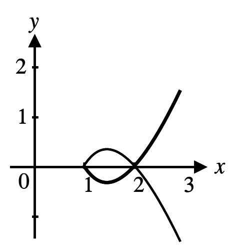


Fig. 6.6

EXERCISE 6.2

(1) Trace the curve $y = x^3$

Discuss the following curves for (i) existence (ii) symmetry (iii) asymptotes (iv) loops

(2) $y^2 = x^2(1 - x^2)$ (3) $y^2(2 + x) = x^2(6 - x)$

(4) $y^2 = x^2(1 - x)$ (5) $y^2 = (x - a)(x - b)^2$; $a, b > 0$, $a > b$.

6.3 Partial Differentiation :

A nation's economy (E) depends on many factors. An yield (Y) of a crop also depends on various factors such as rain, soil, manure etc., Similarly the character (C) of a child is formed by its parent's characters, environment etc., In plane geometry, area (A) and volume (V) also depend on the dimensions like length, breadth and height. In all the above cases either economy or yield or character or area or volume depends on more than one variable (factor). If any small change is effected in any of the variables (factors), it becomes necessary to know what changes will be caused in the respective dependent variable E or Y or C or A or V. These small changes can take place in all the variables (independent) simultaneously or in some of them while others are not subjected to any change. The study of these changes in the dependent variable while a corresponding change is made in one or more of the independent variables, keeping the remaining independent variables fixed leads to what is known as partial differentiation.

For clarity, let us consider the area (A) of a rectangle of length x and breadth y. Then A $= xy = f(x,y)$. Note that 'A' depends on two independent variables x and y.

A $= xy$ = area of *abcd*

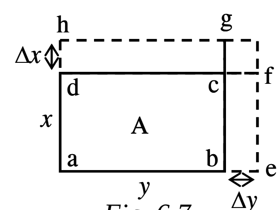


Fig. 6.7

Suppose a small change is made in y *ie.*, $y + \Delta y$ instead of y, then the new area $A' = x(y + \Delta y)$. Note that x is fixed still there is change in the area A. Similarly, if we interchange roles of x and y in the above we get new area *abgh* $= A'' = (x + \Delta x)y$.

Note that change in both x and y will also cause change in area A. In this case the area is $(x + \Delta x)\ (y + \Delta y)$ = area of *aeih*.

But we shall restrict ourselves to the discussion of the change in one variable fixing the rest. We may consider functions of two or three independent variables only.

We can also discuss the continuity problems and the limit process for functions depending on more than one variable similar to that of their counterpart in single variable differential calculus.

Partial Derivatives :

Let (x_0,y_0) be any point in the domain of definition of $f(x,y)$. Let $u = f(x, y)$ We define partial derivative of u with respect to x at the point (x_0,y_0) as the ordinary derivative of $f(x,y_0)$ with respect to x at the point $x = x_0$.

$$i.e., \left.\frac{\partial u}{\partial x}\right]_{(x_0,\, y_0)} = \left.\frac{d}{dx}\, f(x,y_0)\right]_{x = x_0}$$

$$= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0,y_0)}{h}, \text{ (denoted by } f_x \text{ or } u_x \text{ at } (x_0, y_0))$$

provided the limit exists.

Similarly, partial derivatives of $u = f(x,y)$ with respect to y at the point (x_0,y_0) is

$$\left.\frac{\partial u}{\partial y}\right]_{(x_0,\, y_0)} = \left.\frac{d}{dy}\, f(x_0,y)\right]_{y = y_0}$$

$$= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0,y_0)}{h} \text{ (denoted by } f_y \text{ or } u_y \text{ at } (x_0, y_0))$$

provided the limit exists.

A function is said to be **differentiable** at a point (at all points on a domain) if its partial derivatives exist at that point (at all points of a domain). The process of finding partial derivatives is called **partial differentiation**.

Remark :

Throughout we shall consider only continuous functions of two or three variables possessing continuous first order partial derivatives.

Second Order Partial Derivatives : When we differentiate a function $u = f(x,y)$ twice we obtain its second order derivatives, defined by,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \;;\; \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{ and}$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} \text{ denoted respectively}$$

as f_{xx} or u_{xx}, f_{yy} or u_{yy} and $f_{xy} = f_{yx}$ or $u_{xy} = u_{yx}$

Note that since the function and its partial derivaties are continuous the order of differentiation is immaterial (A result due to Euler)

Chain rule (function of a function rule) of two variables :

If $u = f(x,y)$ is differentiable and x and y are differentiable functions of t, then u is a differentiable function of t and

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Tree diagram to remember the chain rule : (2 variables)

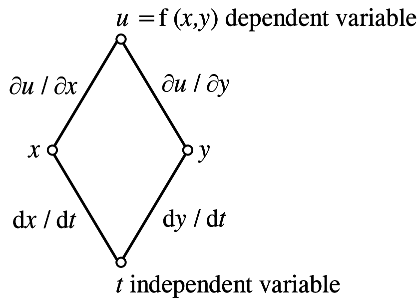


Fig. 6.8

Chain rule (function of a function rule) of three variables :

If $u = f(x,y, z)$ is differentiable and x, y, z are differentiable functions of t, then u is a differentiable function of t and

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Tree diagram to remember the chain rule : (3 – variables)

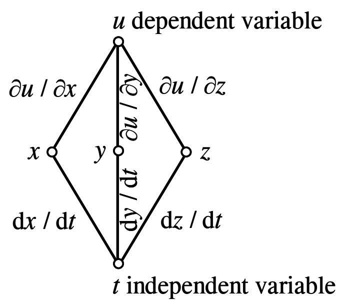


Fig. 6.9

Chain rule for partial derivatives :

If $w = f(u,v)$, $u = g(x,y)$, ; $v = h(x,y)$ then

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \quad ; \quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

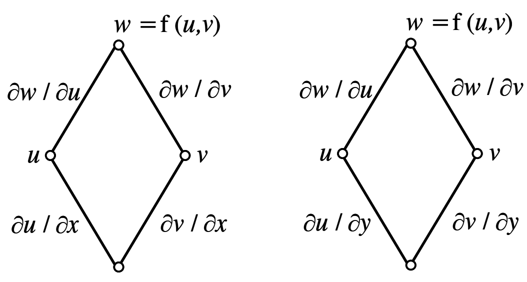


Fig. 6.10

Homogeneous functions :

A function of several variables is said to be homogeneous of degree n if multiplying each variables by t (where $t > 0$) has the same effect as multiplying the original function by t^n. Thus, $f(x,y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x,y)$

Euler's Theorem :

If $f(x,y)$ is a homogeneous function of degree n, then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$

Remark : Euler's theorem can be extended to several variables.

Example 6.14 : Determine : $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}, \dfrac{\partial^2 u}{\partial x\,\partial y}$ and $\dfrac{\partial^2 u}{\partial y\partial x}$

if $u(x,y) = x^4 + y^3 + 3x^2y^2 + 3x^2y$

Solution :

$$\frac{\partial u}{\partial x} = 4x^3 + 6xy^2 + 6xy \;;\; \frac{\partial u}{\partial y} = 3y^2 + 6x^2y + 3x^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 + 6y^2 + 6y \;;\; \frac{\partial^2 u}{\partial y^2} = 6y + 6x^2$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = 12\,xy + 6x \;;\; \frac{\partial^2 u}{\partial y\partial x} = 12xy + 6x$$

Note that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\partial x}$ due to continuity of u and its first order partial derivatives.

Example 6.15 : If $u = \log(\tan x + \tan y + \tan z)$, prove that $\sum \sin 2x\dfrac{\partial u}{\partial x} = 2$

Solution :

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x\frac{\partial u}{\partial x} = \frac{2\sin x\cos x\,.\,\sec^2 x}{\tan x + \tan y + \tan z} = \frac{2\tan x}{\tan x + \tan y + \tan z}$$

similarly, $$\sin 2y\frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z}$$

$$\sin 2z\frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$\text{L.H.S.} = \sum \sin 2x\frac{\partial u}{\partial x} = \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2 = \text{R.H.S}$$

Example 6.16 :

If $U=(x-y)(y-z)(z-x)$ then show that $U_x + U_y + U_z = 0$

Solution : $U_x = (y-z)\{(x-y)(-1) + (z-x).1\}$

$= (y-z)[(z-x)-(x-y)]$

Similarly $U_y = (z-x)[(x-y)-(y-z)]$

$U_z = (x-y)[(y-z)-(z-x)]$

$$U_x + U_y + U_z = (y-z)[(z-x)-(z-x)] + (x-y)[-(y-z)+(y-z)] + (z-x)[(x-y)-(x-y)]$$

$= 0$

Example 6.17 : Suppose that $z = ye^{x^2}$ where $x = 2t$ and $y = 1-t$ then find $\frac{dz}{dt}$

Solution : $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$\frac{\partial z}{\partial x} = ye^{x^2}2x$; $\frac{\partial z}{\partial y} = e^{x^2}$; $\frac{dx}{dt} = 2$; $\frac{dy}{dt} = -1$

$\frac{dz}{dt} = y\,2x\,e^{x^2}(2) + e^{x^2}(-1)$

$= 4xy\,e^{x^2} - e^{x^2} = e^{4t^2}[(8t(1-t)-1)] = e^{4t^2}(8t - 8t^2 - 1)$

(Since $x = 2t$ and $y = 1-t$)

Example 6.18 : If $w = u^2 e^v$ where $u = \frac{x}{y}$ and $v = y \log x$, find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$

Solution : We know $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$; and $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$

$\frac{\partial w}{\partial u} = 2ue^v$; $\frac{\partial w}{\partial v} = u^2e^v$;

$\frac{\partial u}{\partial x} = \frac{1}{y}$; $\frac{\partial u}{\partial y} = \frac{-x}{y^2}$

$\frac{\partial v}{\partial x} = \frac{y}{x}$; $\frac{\partial v}{\partial y} = \log x.$

$\therefore \frac{\partial w}{\partial x} = \frac{2ue^v}{y} + u^2e^v\frac{y}{x} = x^y\frac{x}{y^2}(2+y)$

$$\therefore \ \frac{\partial w}{\partial y} = 2ue^{v}\ \frac{-x}{y^2} + u^2e^{v}\ \log x$$

$$= \frac{x^2}{y^3}\ x^{y}\ [y\log x - 2], \quad (\text{since } u = \frac{x}{y} \text{ and } v = y \log x)$$

Example 6.19 : If $w = x + 2y + z^2$ and $x = \cos t$; $y = \sin t$; $z = t$. Find $\frac{dw}{dt}$

Solution : We know $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$$\frac{\partial w}{\partial x} = 1\ ;\ \frac{dx}{dt} = -\sin t$$

$$\frac{\partial w}{\partial y} = 2\ ;\ \frac{dy}{dt} = \cos t$$

$$\frac{\partial w}{\partial z} = 2\,\mathrm{z}\ ;\ \frac{dz}{dt} = 1$$

$$\therefore \ \frac{dw}{dt} = 1\,(-\sin t) + 2\cos t + 2z = -\sin t + 2\cos t + 2\,t$$

Example 6.20 : Verify Euler's theorem for $f(x,y) = \frac{1}{\sqrt{x^2 + y^2}}$

Solution : $f(tx, ty) = \frac{1}{\sqrt{t^2\,x^2 + t^2y^2}} = \frac{1}{t}\ f(x,y) = t^{-1} f(x, y)$

$\therefore$ f is a homogenous function of degree -1 and by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = -f$$

Verification : $f_x = -\frac{1}{2}\frac{2x}{\left(x^2+y^2\right)^{3/2}} = \frac{-x}{\left(x^2+y^2\right)^{3/2}}$

Similarly, $f_y = \frac{-y}{\left(x^2+y^2\right)^{3/2}}$

$$xf_x + yf_y = -\frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}} = \frac{-1}{\sqrt{x^2+y^2}} = -f.$$

Hence Euler's theorem is verified.

Example 6.21 : If u is a homogenous function of x and y of degree n, prove that

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

Solution : Since U is a homogeneous function in x and y of degree n, U_y is homogeneous function in x and y of degree $n-1$. Applying Euler's theorem for U_y we have,

$$x(U_y)_x + y\,(U_y)_y = (n-1)\,U_y$$

$$\text{i.e.,}\quad xU_{yx} + y\,U_{yy} = (n-1)\,U_y$$

$$\text{i.e.,}\quad x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

Example 6.22 : Using Euler's theorem, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial \mathrm{y}} = \frac{1}{2}\tan u$ if $u = \sin^{-1}\left(\frac{x-y}{\sqrt{x}+\sqrt{y}}\right)$

Solution: R.H.S. is not homogeneous and hence

define $f = \sin u = \frac{x-y}{\sqrt{x}+\sqrt{y}} \Rightarrow f$ is homogeneous of degree $\frac{1}{2}$.

$\therefore$ By Euler's theorem, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f$

i.e., $x\,.\,\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$

$x\frac{\partial u}{\partial x}\,.\,\cos u + y\frac{\partial u}{\partial y}\,.\,\cos u = \frac{1}{2}\sin u$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

EXERCISE 6.3

(1) Verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ for the following functions :

(i) $u = x^2 + 3xy + y^2$

(ii) $u = \frac{x}{y^2} - \frac{y}{x^2}$

(iii) $u = \sin 3x\cos 4y$

(iv) $u = \tan^{-1}\left(\frac{x}{y}\right)$.

(2) (i) If $u = \sqrt{x^2 + y^2}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$

(ii) If $u = e^{\frac{x}{y}} \sin \frac{x}{y} + e^{\frac{y}{x}} \cos \frac{y}{x}$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$.

(3) Using chain rule find $\frac{\mathrm{dw}}{\mathrm{dt}}$ for each of the following :

(i) $w = e^{xy}$ where $x = t^2, \quad y = t^3$

(ii) $w = \log (x^2 + y^2)$ where $x = e^t, \; y = e^{-t}$

(iii) $w = \frac{x}{(x^2 + y^2)}$ where $x = \cos t, \; y = \sin t$.

(iv) $w = xy + z$ where $x = \cos t, \; y = \sin t, z = t$

(4) (i) Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ if $w = \log (x^2 + y^2)$ where $x = r \cos \theta, y = r \sin \theta$

(ii) Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = x^2 + y^2$ where $x = u^2 - v^2, \; y = 2uv$

(iii) Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ if $w = \sin^{-1} xy$ where $x = u + v, \; y = u - v$.

(5) Using Euler's theorem prove the following :

(i) If $u = \tan^{-1} \left(\frac{x^3 + y^3}{x - y} \right)$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$.

(ii) $u = xy^2 \sin \left(\frac{x}{y} \right)$, show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3u$.

(iii) If u is a homogeneous function of x and y of degree n, prove that $x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \, \partial y} = (n - 1) \frac{\partial u}{\partial x}$

(iv) If $\mathrm{V} = ze^{ax + by}$ and z is a homogenous function of degree n in x and y prove that $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} = (ax + by + n)\mathrm{V}$.

7. INTEGRAL CALCULUS AND ITS APPLICATIONS

7.1. Introduction :

In class XI, we have studied the direct evaluation of definite integrals as the limit of integral sums. Even when the integrands are very simple, direct evaluation of definite integrals as the limit of integral sum involves great difficulties. Sometimes this method involves cumbersome computations. There is a formula called Second Fundamental Theorem on Calculus that yields a practical and convenient method for computing definite integrals in case where the anti-derivative of the integrand is known. This method which was discovered by Newton and Leibnitz utilises 'the profound relationship' that exists between integration and differentiation. In this chapter we have the following five sections dealing with the concept and applications of definite integrals.

(i) To solve simple problems using second fundamental theorem of calculus.
(ii) Properties of definite integral.
(iii) Reduction formulae
(iv) Area under the curve and volume of solid of revolution about an axis.
(v) Length of the curve and the surface area of a solid of revolution about an axis.

7.2. Simple definite integrals :

First fundamental theorem of calculus :

Theorem 7.1 : If $f(x)$ is a continuous function and $F(x) = \int\limits_a^x f(t)dt$, then we have the equation $F'(x) = f(x)$.

Second fundamental theorem of calculus :

Theorem 7.2 : If $f(x)$ is a continuous function with domain $a \le x \le b$, then

$\int\limits_a^b f(x)dx = F(b) - F(a)$ where F is any anti-derivative of f.

Example 7.1 : Evaluate $\int\limits_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$

Solution: Let $I = \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\, dx$

Let $t = \cos x$

$dt = -\sin x\ dx$ (or) $\sin x\, dx = -dt$

$t = \cos x$

x	0	$\pi / 2$
t	1	0

$\therefore I = \int_1^0 \frac{-dt}{1+t^2} = -\left[\tan^{-1} t\right]_1^0 = -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$

Example 7.2 : Evaluate $\int_0^1 x\, e^x\, dx$

Solution:

Using the method of integration by parts $\left(\int u dv = uv - \int v\, du\right)$

Here $u = x$

$du = dx$

$dv = e^x\, dx$

$v = e^x$

$$\int_0^1 x\, e^x\, dx = (xe^x)_0^1 - \int_0^1 e^x\, dx$$

$$= e - (e^x)_0^1$$

$$= e - (e - 1)$$

$$= 1$$

Example 7.3 : Evaluate $\int_0^a \sqrt{a^2 - x^2}\, dx$

Solution: $\int_0^a \sqrt{a^2 - x^2}\, dx = \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$$= \left[0 + \frac{a^2}{2}\sin^{-1}\frac{a}{a} - (0 + 0)\right]$$

$$= \frac{a^2}{2}\sin^{-1}(1) = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

Example 7.4 : Evaluate $\int_0^{\pi/2} e^{2x} \cos x \, dx$

Solution: We know $\int e^{ax} \cos bx \, dx = \left(\frac{e^{ax}}{a^2 + b^2}\right)(a \cos bx + b \sin bx)$

$$\therefore \int_0^{\pi/2} e^{2x} \cos x \, dx = \left[\left(\frac{e^{2x}}{2^2 + 1^2}\right)(2 \cos x + \sin x)\right]_0^{\pi/2}$$

$$= \frac{e^{\pi}}{5}(0 + 1) - \frac{e^0}{5}(2 + 0)$$

$$= \frac{e^{\pi}}{5} - \frac{2}{5} = \frac{1}{5}(e^{\pi} - 2)$$

EXERCISE 7.1

Evaluate the following problems using second fundamental theorem :

(1) $\int_0^{\pi/2} \sin^2 x \, dx$ (2) $\int_0^{\pi/2} \cos^3 x \, dx$ (3) $\int_0^1 \sqrt{9 - 4x^2} \, dx$

(4) $\int_0^{\pi/4} 2 \sin^2 x \sin 2x \, dx$ (5) $\int_0^1 \frac{dx}{\sqrt{4 - x^2}}$ (6) $\int_0^{\pi/2} \frac{\sin x \, dx}{9 + \cos^2 x}$

(7) $\int_1^2 \frac{dx}{x^2 + 5x + 6}$ (8) $\int_0^1 \frac{(\sin^{-1} x)^3}{\sqrt{1 - x^2}} \, dx$ (9) $\int_0^{\pi/2} \sin 2x \cos x \, dx$

(10) $\int_0^1 x^2 e^x \, dx$ (11) $\int_0^{\pi/2} e^{3x} \cos x \, dx$ (12) $\int_0^{\pi/2} e^{-x} \sin x \, dx$

7.3 Properties of Definite Integrals :

Property (1) : $\boxed{\int_a^b f(x) dx = \int_a^b f(y) \, dy}$

Proof : Let F be any anti-derivative of f

$$\therefore \int_a^b f(x) \, dx = [F(b) - F(a)] \qquad \ldots \text{(i)}$$

$$\int_a^b f(y)\, dy = [F(b) - F(a)] \qquad \dots \text{(ii)}$$

From (i) and (ii) $\quad \int_a^b f(x)\, dx = \int_a^b f(y)\, dy$

That is, integration is independent of change of variables provided the limits of integration remain the same.

Property (2) : $\boxed{\int_a^b f(x)dx = -\int_b^a f(x)\, dx}$

Proof : Let F be any anti-derivative of f

$$\therefore \int_a^b f(x)\, dx = [F(b) - F(a)] \qquad \dots \text{(i)}$$

$$\int_b^a f(x)\, dx = [F(a) - F(b)] = -[F(b) - F(a)] \qquad \dots \text{(ii)}$$

From (i) and (ii) $\quad \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

That is, if the limits of definite integral are interchanged, then the value of integral changes its sign only.

Property (3) : $\int_a^b f(x)dx = \int_a^b f(a + b - x)\, dx$

Proof : Let $u = a + b - x$

$\therefore \; du = -dx$

or $dx = -du$

$u = a + b - x$

x	a	b
u	b	a

$$\therefore \int_a^b f(a + b - x)dx = -\int_b^a f(u)\, du = \int_a^b f(u)\, du = \int_a^b f(x)\, dx$$

Property (4) : $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$

Proof : Let $u = a - x$

$\therefore\ du = -dx$

or $dx = -du$

$u = a - x$

x	o	a
u	a	o

$$\therefore \int_0^a f(a-x)dx = -\int_a^o f(u)\,du = \int_0^a f(u)\,du = \int_0^a f(x)\,dx$$

Property (5) (Without proof) : If $f(x)$ is integrable on a closed interval containing the three numbers a, b and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

regardless of the order of a, b and c.

Property (6) : $\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Proof : Consider $\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$ … (1)

Put $x = 2a - u$ in the second integral on the R.H.S.,

and $dx = -du$

$u = 2a - x$

x	a	$2a$
u	a	o

$$\int_a^{2a} f(x)dx = -\int_a^o f(2a-u)\,du$$

$$= \int_0^a f(2a-u)\,du$$

$$= \int_0^a f(2a-x)\,dx \qquad \left(\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right)$$

Hence (1) becomes $\int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_0^{a} f(2a - x)\, dx$

Property (7) : $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)\, dx \qquad$ if $f(2a - x) = f(x)$

$= 0 \qquad$ if $f(2a - x) = - f(x)$

Proof : We know that by property

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)\, dx + \int_0^{a} f(2a - x)\, dx \qquad \ldots (1)$$

If $f(2a - x) = f(x)$ then (1) becomes

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)\, dx + \int_0^{a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$$

If $f(2a - x) = - f(x)$ then (1) becomes

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)\, dx - \int_0^{a} f(x)\, dx = 0$$

Hence proved.

Property (8) : (i) $\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)\, dx, \qquad$ if f is an even function.

(ii) $\int_{-a}^{a} f(x)\, dx = 0 \qquad$ if f is an odd function.

Proof : Consider $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)\, dx + \int_0^{a} f(x)\, dx \qquad \ldots (1)$

Let $\quad x = - t$ in the first integral of the R.H.S.

Then $dx = - dt$

$x = - t$

x	$-a$	0
t	a	0

$\therefore$ (1) becomes $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{a}^{o} f(-t)\,(-dt) + \int_{0}^{a} f(x)\,dx$

$$= -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \ldots (2)$$

Case (ii) : If '*f*' is an even function, then (2) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

Case (iii) : If '*f*' is an odd function then (2) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} (-f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Hence proved.

Example 7.5 : Evaluate $\displaystyle\int_{-\pi/4}^{\pi/4} x^3 \sin^2 x\,dx$.

Solution: Let $f(x) = x^3 \sin^2 x = x^3 (\sin x)^2$

$$\therefore f(-x) = (-x)^3 (\sin(-x))^2$$

$$= (-x)^3 (-\sin x)^2$$

$$= -x^3 \sin^2 x$$

$$= -f(x)$$

$$f(-x) = -f(x)$$

$\therefore$ $f(x)$ is an odd function.

$$\therefore \int_{-\pi/4}^{\pi/4} x^3 \sin^2 x \, dx. = 0 \text{ (by property)}$$

Example 7.6 :

Evaluate $\displaystyle\int_{-1}^{1} \log\left(\frac{3-x}{3+x}\right) dx$

Solution: Let $f(x) = \log\left(\dfrac{3-x}{3+x}\right)$

$$\therefore f(-x) = \log\left(\frac{3+x}{3-x}\right) = \log(3+x) - \log(3-x)$$

$$= -\left[\log(3-x) - \log(3+x)\right]$$

$$= -\left[\log\left(\frac{3-x}{3+x}\right)\right] = -f(x)$$

Thus $f(-x) = -f(x)$ $\therefore$ $f(x)$ is an odd function.

$$\therefore \int_{-1}^{1} \log\left(\frac{3-x}{3+x}\right) dx = 0$$

Example 7.7 :

Evaluate : $\displaystyle\int_{-\pi/2}^{\pi/2} x \sin x \, dx$

Solution: Let $f(x) = x \sin x$

$$f(-x) = (-x)\sin(-x)$$

$$= x \sin x \;(\because \sin(-x) = -\sin x)$$

$\therefore$ $f(x)$ is an even function.

$$\int_{-\pi/2}^{\pi/2} x \sin x \, dx = 2 \int_{0}^{\pi/2} x \sin x \, dx$$

$$= 2\left[\left\{x(-\cos x)\right\}_{0}^{\pi/2} - \int_{0}^{\pi/2} (-\cos x) \, dx\right]$$

Using the method of integration by parts

$$= 2\left[0 + \int_0^{\pi/2} \cos x\, dx\right] = 2\,[\sin x]_0^{\pi/2}$$

$$= 2\,[1 - 0] = 2$$

Example 7.8 : Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$

Solution: Let $f(x) = \sin^2 x = (\sin x)^2$

$$f(-x) = (\sin(-x))^2 = (-\sin x)^2 = \sin^2 x = f(x)$$

Hence $f(x)$ is an even function.

$$\therefore \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx = 2\int_0^{\pi/2} \sin^2 x\, dx = 2 \times \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x)\, dx$$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$$

Example 7.9 : Evaluate $\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)}\, dx$

Solution: Let $I = \int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)}\, dx \qquad \ldots (1)$

$$= \int_0^{\pi/2} \frac{f\left(\sin\left(\frac{\pi}{2} - x\right)\right)}{f\left(\sin\left(\frac{\pi}{2} - x\right)\right) + f\left(\cos\left(\frac{\pi}{2} - x\right)\right)}\, dx$$

$$\therefore I = \int_0^{\pi/2} \frac{f(\cos x)}{f(\cos x) + f(\sin x)}\, dx \qquad \ldots (2)$$

(1) + (2) gives $2I = \int_o^{\pi/2} \frac{f(\sin x) + f(\cos x)}{f(\cos x) + f(\sin x)}\, dx = \int_o^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

$$\therefore I = \frac{\pi}{4}$$

Example 7.10 : Evaluate $\int_0^1 x(1-x)^n \, dx$

Solution: Let $I = \int_0^1 x(1-x)^n \, dx$

$$= \int_0^1 (1-x)\left[1-(1-x)\right]^n dx \qquad \left[\because \int_0^a f(x)\,dx = \int_o^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)\,x^n\,dx = \int_0^1 (x^n - x^{n+1})\,dx$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \left[\frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{n+2-(n+1)}{(n+1)(n+2)}$$

$$\int_0^1 x(1-x)^n\,dx = \frac{1}{(n+1)(n+2)}$$

Example 7.11 : Evaluate $\int_0^{\pi/2} \log(\tan x)dx$

Solution: Let $I = \int_0^{\pi/2} \log(\tan x)dx \qquad \ldots (1)$

$$= \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx$$

$$I = \int_0^{\pi/2} \log(\cot x)\,dx \qquad \ldots (2)$$

(1) + (2) gives $\quad 2I = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\,dx$

$$= \int_0^{\pi/2} [\log (\tan x) . (\cot x)]\, dx = \int_0^{\pi/2} (\log 1)\, dx = 0$$

$$\therefore\ I = 0 \qquad (\because \log 1 = 0)$$

Example 7.12 : Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

Solution: Let $\displaystyle I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \qquad \ldots (1)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}\, dx}{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}$$

$$\left(\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \qquad \ldots (2)$$

(1) + (2) gives $\displaystyle 2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\ dx$

$$2I = \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore\ I = \frac{\pi}{12}$$

EXERCISE 7.2

Evaluate the following problems using properties of integration.

(1) $\int_{-1}^{1} \sin x \cos^4 x\, dx$ (2) $\int_{-\pi/4}^{\pi/4} x^3 \cos^3 x\, dx$ (3) $\int_{0}^{\pi/2} \sin^3 x \cos x\, dx$

(4) $\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx$ (5) $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$ (6) $\int_{-\pi/4}^{\pi/4} x \sin^2 x\, dx$

(7) $\int_{0}^{1} \log\left(\frac{1}{x} - 1\right) dx$ (8) $\int_{0}^{3} \frac{\sqrt{x}\, dx}{\sqrt{x} + \sqrt{3 - x}}$ (9) $\int_{0}^{1} x (1 - x)^{10}\, dx$

(10) $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$

7.4 Reduction formulae :

A formula which expresses (or reduces) the integral of the nth indexed function interms of that of $(n - 1)$th indexed (or lower indexed) function is called a reduction formula.

Reduction formulae for $\int \sin^n x\, dx$. $\int \cos^n x\, dx$ (n is a positive integer) :

Result 1 : If $I_n = \int \sin^n x\, dx$ then $I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$

Result 2 : If $I_n = \int \cos^n x\, dx$ then $I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$

Result 3 :

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1 \text{ when } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \text{ when } n \text{ is even} \end{cases}$$

Note : For the proofs of these above three results, refer Solution Book.

Example 7.13 : Evaluate : $\int \sin^5 x \, dx$

Solution : If $I_n = \int \sin^n x \, dx$, then we have

$$I_n = -\frac{1}{n}\sin^{n-1}x \cos x + \frac{n-1}{n} I_{n-2} \qquad \ldots \text{(I)}$$

$$\therefore \int \sin^5 x \, dx = I_5$$

$$= -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5} I_3 \qquad \text{(when } n\text{=5 in I)}$$

$$= -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}\left[-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3} I_1\right] \text{(when } n\text{=3 in I)}$$

$$\int \sin^5 x \, dx = -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x + \frac{8}{15} I_1 \qquad \ldots \text{(II)}$$

$$I_1 = \int \sin^1 x \, dx = -\cos x + c$$

$$\therefore \int \sin^5 x \, dx = -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + c$$

Example 7.14 : Evaluate : $\int \sin^6 x \, dx$

Solution : If $I_n = \int \sin^n x \, dx$, then we have

$$I_n = -\frac{1}{n}\sin^{n-1}x \cos x + \frac{n-1}{n} I_{n-2} \qquad \ldots \text{(I)}$$

$$\therefore \int \sin^6 x \, dx = I_6$$

$$= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6} I_4 \qquad \text{(when } n\text{=6 in I)}$$

$$= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\left[-\frac{1}{4}\sin^3 x \cos x + \frac{3}{4} I_2\right] \qquad \text{(when } n\text{=4 in I)}$$

$$\int \sin^6 x \, dx = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x + \frac{5}{8} I_2 \qquad \text{(when } n\text{=2 in I)}$$

$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x + \frac{5}{8}\left[-\frac{1}{2}\sin x \cos x + \frac{1}{2} I_0\right]$$

$$\int \sin^6 x \, dx = -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5}{16} I_0$$

$$I_0 = \int \sin^0 x\,dx = \int dx = x$$

$$\therefore \int \sin^6 x\,dx = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x$$

Example 7.15 : Evaluate :

(i) $\int_0^{\pi/2} \sin^7 x\,dx$ (ii) $\int_0^{\pi/2} \cos^8 x\,dx$ (iii) $\int_0^{2\pi} \sin^9 \frac{x}{4}\,dx$

(iv) $\int_0^{\pi/6} \cos^7 3x\,dx$

Solution : (i)We have

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} \text{ when `}n\text{' is odd}$$

$$\int_0^{\pi/2} \sin^7 x\,dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{16}{35}$$

(ii) $$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \text{ when `}n\text{' is even}$$

$$\therefore \int_0^{\pi/2} \cos^8 x\,dx = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35\pi}{256}$$

(iii) $$\int_0^{2\pi} \sin^9 \frac{x}{4}\,dx$$

Put $\frac{x}{4} = t$

$\therefore\ dx = 4dt$

$t = x / 4$

x	0	2π
t	0	$\pi/2$

$$\int_0^{2\pi} \sin^9 \frac{x}{4}\,dx = 4\int_0^{\pi/2} \sin^9 t\,dt = 4.\left(\frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot\right) = \frac{512}{315}$$

(iv) $\int_0^{\pi/6} \cos^7 3x \, dx$

Put $3x = t$

$3dx = dt$

$dx = 1/3 \, dt$

$t = 3x$

x	0	$\pi/6$
t	0	$\pi/2$

$$\int_0^{\pi/6} \cos^7 3x \, dx = \frac{1}{3} \int_0^{\pi/2} \cos^7 t \, dt = \frac{1}{3}\left[\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot\right] = \frac{16}{105}$$

Example 7.16 : Evaluate : $\int_0^{\pi/2} \sin^4 x \cos^2 x \; dx$

Solution :

$$\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx = \int_0^{\pi/2} \sin^4 x \,(1 - \sin^2 x)\, dx$$

$$= \int_0^{\pi/2} (\sin^4 x - \sin^6 x)\, dx = \int_0^{\pi/2} \sin^4 x \, dx - \int_0^{\pi/2} \sin^6 x \, dx$$

$$= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{32}$$

Two important results : The following two results are very useful in the evaluation of certain types of integrals.

(1) If u and v are functions of x, then

$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + ... + (-1)^n u^n v_n + ...$

where u', u'', u''' ... are successive derivatives of u and v_1, v_2, v_3 ... are repeated integrals of v

The above formula is well known as **Bernoulli's** formula.

Bernoulli's formula is advantageously applied when $u = x^n$ (n is a positive integer).

(2) If n is a positive integer, then $\int_0^{\infty} x^n e^{-ax} \, dx = \frac{\lfloor n}{a^{n+1}}$

Note : The above formula is known as a particular case of Gamma Integral.

Example 7.17 : Evaluate :

(i) $\int x^3 e^{2x}\, dx$ (ii) $\int_0^1 x\, e^{-4x}\, dx$ (iii) $\int_0^\infty x^5 e^{-4x}\, dx$ (iv) $\int_0^\infty e^{-mx} x^7\, dx$

Solution :

(1) $\int x^3 e^{2x}\, dx$

Using Bernoulli's formula

$\int u dv = uv - u'v_1 + u''v_2 \ldots$

We get

$$dv = e^{2x}\, dx$$

$$u = x^3 \qquad v = 1/2\, e^{2x}$$

$$u' = 3x^2 \qquad v_1 = 1/4\, e^{2x}$$

$$u'' = 6x \qquad v_2 = 1/8\, e^{2x}$$

$$u''' = 6 \qquad v_3 = 1/16\, e^{2x}$$

$$\int x^3 e^{2x}\, dx = (x^3)\left(\frac{1}{2} e^{2x}\right) - (3x^2)\left(\frac{1}{4} e^{2x}\right) + (6x)\left(\frac{1}{8} e^{2x}\right) - (6)\left(\frac{1}{16} e^{2x}\right)$$

$$= \frac{1}{2} e^{2x}\left[x^3 - \frac{3}{2}x^2 + \frac{3x}{2} - \frac{3}{4}\right]$$

(ii) $\int_0^1 x\, e^{-4x}\, dx$

$$dv = e^{-4x}\, dx$$

$$u = x \qquad v = -\frac{1}{4} e^{-4x}$$

$$u' = 1 \qquad v_1 = \frac{1}{16} e^{-4x}$$

Using Bernoulli's formula we get

$$\int_0^1 x\, e^{-4x}\, dx = \left[(x)\left(-\frac{1}{4} e^{-4x}\right) - (1)\left(\frac{1}{16} e^{-4x}\right)\right]_0^1$$

$$= \left(-\frac{1}{4} e^{-4} - 0\right) - \frac{1}{16}(e^{-4} - e^0)$$

$$= \frac{1}{16} - \frac{5}{16} e^{-4}$$

(iii) $\int_0^\infty x^5 e^{-4x}\, dx$ Using Gamma Integral $\int_0^\infty x^5 e^{-4x}\, dx = \frac{\lfloor 5}{4^6}$

(iv) $\int_0^\infty e^{-mx} x^7\, dx = \frac{\lfloor 7}{m^8}$ (Using Gamma Integral)

EXERCISE 7.3

(1) Evaluate : (i) $\int \sin^4 x \, dx$ (ii) $\int \cos^5 x \, dx$

(2) Evaluate : (i) $\int_0^{\pi/2} \sin^6 x \, dx$ (ii) $\int_0^{\pi/2} \cos^9 x \, dx$

(3) Evaluate : (i) $\int_0^{\pi/4} \cos^8 2x \, dx$ (ii) $\int_0^{\pi/6} \sin^7 3x \, dx$

(4) Evaluate : (i) $\int_0^{1} x\, e^{-2x} \, dx$ (ii) $\int_0^{\infty} x^6\, e^{-x/2} \, dx$

7.5 Area and Volume :

In this section, we apply the definite integral to compute measure of area, length of arc and surface area. In our treatment it is understood that area, volume etc. is a number without any unit of measurement attached to it.

7.5.1 Area of bounded regions :

Theorem : Let $y = f(x)$ be a continuous function defined on $[a, b]$, which is positive ($f(x)$ lies on or above x-axis) on the interval $[a, b]$. Then, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by

$$\boxed{\text{Area} = \int_a^b f(x)dx \text{ or } \int_a^b y\,dx}$$

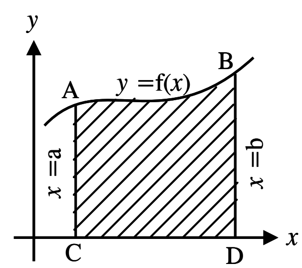


Fig. 7.1

If $f(x) \leq 0$ ($f(x)$ lies on or below x-axis) for all x in $a \leq x \leq b$ then area is given by

$$\boxed{\text{Area} = \int_a^b (-\,y)\,dx = \int_a^b (-\,f(x)\,dx)}$$

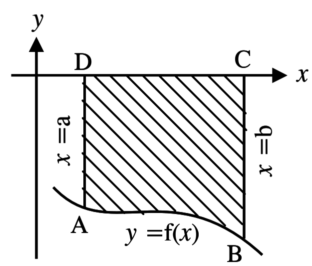


Fig. 7.2

(i.e., The area below the x-axis is negative)

Example 7.18 : Find the area of the region bounded by the line $3x - 2y + 6 = 0$, $x = 1$, $x = 3$ and x-axis.

Since the line $3x - 2y + 6 = 0$ lies above the x-axis in the interval [1, 3],

(i.e., $y > 0$ for $x \in (1,3)$)

the required area

$$A = \int_1^3 y\,dx = \frac{3}{2}\int_1^3 (x+2)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} + 2x\right]_1^3$$

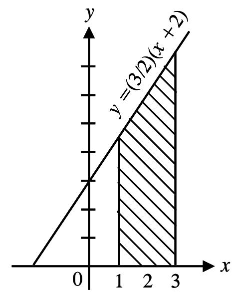


Fig. 7.3

$$= \frac{3}{2}\left[\frac{1}{2}(9-1) + 2(3-1)\right] = \frac{3}{2}[4+4]$$

Area = 12 sq. units

Example 7.19:

Find the area of the region bounded by the line $3x - 5y - 15 = 0$, $x = 1$, $x = 4$ and x-axis.

The line $3x - 5y - 15 = 0$ lies below the x-axis in the interval $x = 1$ and $x = 4$

$$\therefore \text{Required area } = \int_1^4 (-y)\,dx$$

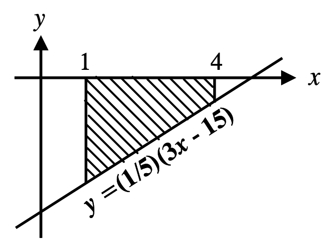


Fig. 7.4

$$= \int_1^4 -\frac{1}{5}(3x-15)\,dx = \frac{3}{5}\int_1^4 (5-x)\,dx = \frac{3}{5}\left[5x - \frac{x^2}{2}\right]_1^4$$

$$= \frac{3}{5}\left[5(4-1) - \frac{1}{2}(16-1)\right]$$

$$= \frac{3}{5}\left[15 - \frac{15}{2}\right] = \frac{9}{2} \text{ sq. units.}$$

Example 7.20: Find the area of the region bounded $y = x^2 - 5x + 4$, $x = 2$, $x = 3$ and the x-axis.

For all x, $2 \le x \le 3$ the curve lies below the x-axis.

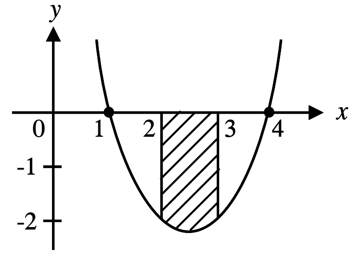


Fig. 7.5

$$\text{Required area} = \int_2^3 (-y)\, dx$$

$$= \int_2^3 -(x^2 - 5x + 4)\, dx$$

$$= -\left[\frac{x^3}{3} - 5\frac{x^2}{2} + 4x\right]_2^3$$

$$= -\left[\left(9 - \frac{45}{2} + 12\right) - \left(\frac{8}{3} - \frac{20}{2} + 8\right)\right] = -\left[\frac{-13}{6}\right] = \frac{13}{6} \text{ sq. units}$$

Area between a continuous curve and *y*-axis :

Let $x = f(y)$ be a continuous function of y on $[c, d]$. The area bounded by the curve $x = f(y)$ and the abscissae $y = c$, $y = d$ to the right of y-axis is given by $\int_c^d x dy$

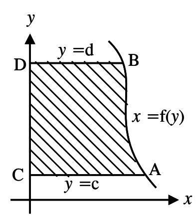


Fig. 7.6

If the curve lies to the left of y-axis between the lines $y = c$ and $y = d$, the area is given by $\int_c^d (-x)\, dy$.

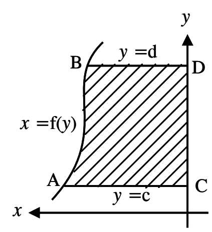


Fig. 7.7

Example 7.21: Find the area of the region bounded by $y = 2x + 1$, $y = 3$, $y = 5$ and y – axis.

Solution : The line $y = 2x + 1$ lies to the right of y-axis between the lines $y = 3$ and $y = 5$.

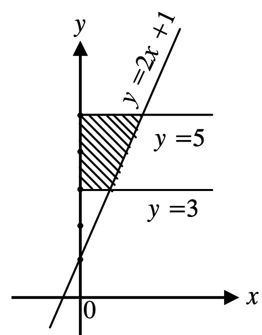


Fig. 7.8

$$\therefore \text{ The required area } A = \int_c^d x dy$$

$$= \int_3^5 \frac{y-1}{2}\,dy = \frac{1}{2}\int_3^5 (y-1)dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - y\right]_3^5 = \frac{1}{2}\left[\left(\frac{25}{2} - \frac{9}{2}\right) - (5-3)\right]$$

$$= \frac{1}{2}\,[8-2] = 3 \text{ sq. units}$$

Example 7.22: Find the area of the region bounded $y = 2x + 4$, $y = 1$ and $y = 3$ and y-axis.

The curve lies to the left of y-axis between the lines $y = 1$ and $y = 3$

$\therefore$ Area is given by

$$A = \int_1^3 (-x)\,dy$$

$$= \int_1^3 -\left(\frac{y-4}{2}\right) dy$$

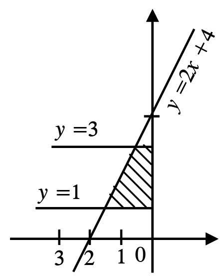


Fig. 7.9

$$= \frac{1}{2}\int_1^3 (4-y)dy = \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_1^3 = \frac{1}{2}\,[8-4] = 2 \text{ sq. units.}$$

Remark :

If the continuous curve f crosses the x-axis, then the integral $\int_a^b \text{f(x)}\,dx$ gives the algebraic sum of the areas between the curve and the axis, counting area above as positive and below as negative.

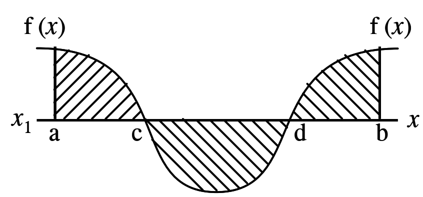


Fig. 7.10

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{\text{above axis}} + \underbrace{\int_c^d (-f(x))\,dx}_{\text{below axis}} + \underbrace{\int_d^b f(x)\,dx}_{\text{above axis}}$$

Example 7.23: (i) Evaluate the integral $\int_1^5 (x-3)dx$

(ii) Find the area of the region bounded by the line $y + 3 = x$, $x = 1$ and $x = 5$

Solution :

(i) $$\int_1^5 (x-3)\,dx = \left[\frac{x^2}{2} - 3x\right]_1^5 = \left(\frac{25}{2} - 15\right) - \left(\frac{1}{2} - 3\right) = 12 - 12 = 0 \ \ldots \ \text{I}$$

(ii) The line $y = x - 3$ crosses x-axis at $x = 3$

From the diagram it is clear that A_1 lies below x-axis.

$$\therefore A_1 = \int_1^3 (-y)\,dx.$$

As A_2 lies above the x-axis

$$A_2 = \int_3^5 y\,dx$$

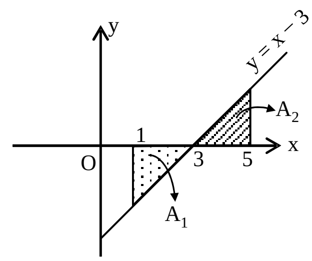


Fig. 7.11

$$\therefore \text{Total area } = \int_1^5 (x-3)dx = \int_1^3 -(x-3)\,dx + \int_3^5 (x-3)\,dx$$

$$= (6-4) + (8-6)$$

$$= 2 + 2$$

$$= 4 \text{ sq. units} \quad \ldots \text{(II)}$$

Note :

From I and II it is clear that the integral $f(x)$ is not always imply an area. The fundamental theorem asserts that the anti-derivative method works even when the function $f(x)$ is not always positive.

Example 7.24:

Find the area bounded by the curve $y = \sin 2x$ between the ordinates $x = 0$, $x = \pi$ and x-axis.

Solution :

The points where the curve $y = \sin 2x$ meets the x-axis can be obtained by putting $y = 0$.

$$\sin 2x = 0 \ \Rightarrow \ 2x = n\pi\ , \ n \in Z$$

$$x = \frac{n}{2}\pi. \qquad \text{i.e., } x = \left\{0, \pm\frac{\pi}{2}, \pm\pi, \pm 3\frac{\pi}{2} \ldots\right\}$$

$\therefore$ The values of x between $x = 0$ are $x = \pi$ are $x = 0, \frac{\pi}{2}, \pi$

The limits for the first arch are 0 and $\frac{\pi}{2}$ and the curve lies above x-axis.

The limits for the second arch are $\frac{\pi}{2}$ and π and the curve lies below x-axis.

$\therefore$ Required area

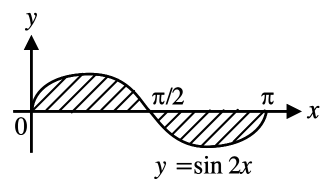


Fig. 7.12

$$A = \int_0^{\pi/2} \sin 2x \, dx + \int_{\pi/2}^{\pi} (-\sin 2x) dx$$

$$= \left(\frac{-\cos 2x}{2}\right)_0^{\pi/2} + \left(\frac{\cos 2x}{2}\right)_{\pi/2}^{\pi}$$

$$= \frac{1}{2}[-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi]$$

$$= \frac{1}{2}[1 + 1 + 1 + 1] = 2 \text{ sq. units.}$$

Example 7.25:

Find the area between the curves $y = x^2 - x - 2$, x-axis and the lines $x = -2$ and $x = 4$

Solution : $y = x^2 - x - 2$

$= (x + 1)(x - 2)$

This curve intersects x-axis at $x = -1$ and $x = 2$

Required area = $A_1 + A_2 + A_3$

The part A_2 lies below x-axis.

$$\therefore A_2 = -\int_{-1}^{2} y \, dx$$

Hence required area

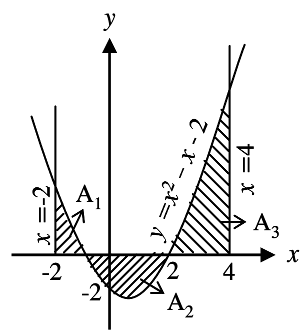


Fig. 7.13

$$= \int_{-2}^{-1} y\,dx + \int_{-1}^{2} (-y)dx + \int_{2}^{4} y\,dx$$

$$= \int_{-2}^{-1} (x^2 - x - 2)\,dx + \int_{-1}^{2} -(x^2 - x - 2)dx + \int_{2}^{4} (x^2 - x - 2)\,dx$$

$$= \frac{11}{6} + \frac{9}{2} + \frac{26}{3} = 15 \text{ sq. units}$$

General Area Principle :

Let f and g be two continuous curves, with f lying above g. then the area R between f and g, from $x = a$ to $x = b$, is given by

$$R = \int_{a}^{b} (f - g)dx$$

No restriction on f and g where they lie. Both may be lie above or below the x-axis or g lies below and f lies above the x-axis.

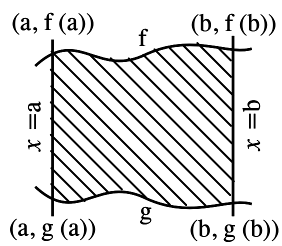


Fig. 7.14

Example 7.26: Find the area between the line $y=x + 1$ and the curve $y = x^2 - 1$.

Solution : To get the points of intersection of the curves we should solve the equations $y = x + 1$ and $y = x^2 - 1$.

we get, $x^2 - 1 = x + 1$

$x^2 - x - 2 = 0$

$\Rightarrow (x - 2)(x + 1) = 0$

$\therefore x = -1$ or $x = 2$

$\therefore$ The line intersects the curve at $x = -1$ and $x = 2$.

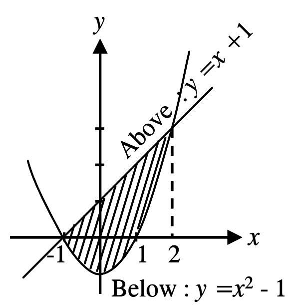


Fig. 7.15

$$\text{Required area} = \int_{a}^{b} \left[\underset{\text{above}}{f(x)} - \underset{\text{below}}{g(x)}\right] dx$$

$$= \int_{-1}^{2} \left[(x + 1) - (x^2 - 1)\right]dx$$

$$= \int_{-1}^{2} [2 + x - x^2]dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[4 + 2 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] = \frac{9}{2} \text{ sq. units}$$

Example 7.27:

Find the area bounded by the curve $y = x^3$ and the line $y = x$.

Solution : The line $y = x$ lies above the curve $y = x^3$ in the first quadrant and $y = x^3$ lies above the line $y = x$ in the third quadrant. To get the points of intersection, solve the curves $y = x^3$, $y = x \Rightarrow x^3 = x$. We get $x = \{0, \pm 1\}$

The required area = $A_1 + A_2 = \int_{-1}^{0} [g(x) - f(x)]dx + \int_{0}^{1} [f(x) - g(x)]dx$

$$= \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} (x - x^3)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

$$= \left(0 - \frac{1}{4}\right) - \left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) - \left(\frac{1}{4} - 0\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2} \text{ sq. units.}$$

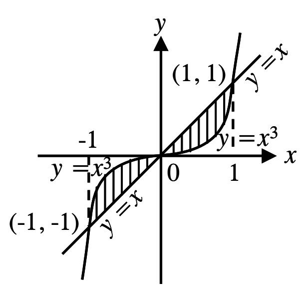


Fig. 7.16

Example 7.28: Find the area of the region enclosed by $y^2 = x$ and $y = x - 2$

Solution : The points of intersection of the parabola $y^2 = x$ and the line $y = x - 2$ are $(1, -1)$ and $(4, 2)$

To compute the region [shown in figure (6.17)] by integrating with respect to x, we would have to split the region into two parts, because the equation of the lower boundary changes at $x = 1$. However if we integrate with respect to y no splitting is necessary.

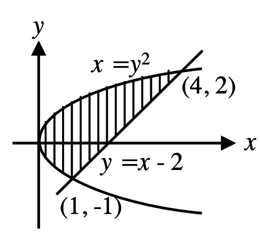


Fig. 7.17

$$\text{Required area} = \int_{-1}^{2} (f(y) - g(y))\, dy$$

$$= \int_{-1}^{2} \left[(y+2) - y^2\right] dy = \left(\frac{y^2}{2} + 2y - \frac{y^3}{3}\right)_{-1}^{2}$$

$$= \left(\frac{4}{2} - \frac{1}{2}\right) + (4+2) - \left(\frac{8}{3} + \frac{1}{3}\right)$$

$$= \frac{3}{2} + 6 - \frac{9}{3} = \frac{9}{2} \text{ sq. units.}$$

Example 7.29: Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$

Solution : The points of intersection of $x^2 + y^2 = 16$ and $y^2 = 6x$ are $(2, 2\sqrt{3})$ and $(2, -2\sqrt{3})$

Required area is *OABC*

Due to symmetrical property, the required area

$$OABC = 2\ OBC$$

i.e., 2{[Area bounded by $y^2 = 6x$, $x = 0$, $x = 2$ and x-axis] + [Area bounded by $x^2 + y^2 = 16$, $x = 2$, $x = 4$ and x-axis]}

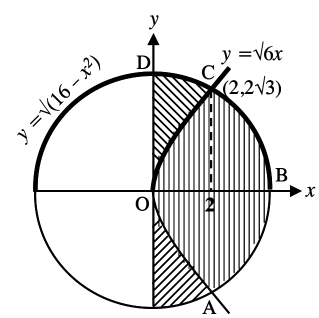


Fig. 7.18

$$= 2\int_{0}^{2} \sqrt{6x}\, dx + 2\int_{2}^{4} \sqrt{16 - x^2}\, dx$$

$$= 2\sqrt{6}\left[\frac{x^{3/2}}{3/2}\right]_0^2 + 2\left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= \frac{8\sqrt{12}}{3} - 2\sqrt{12} + 8\pi - \frac{8\pi}{3}$$

$$= \frac{4}{3}\left(4\pi + \sqrt{3}\right)$$

Example 7.30: Compute the area between the curve $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$

Solution : To find the points of intersection solve the two equations.

$$\text{Sin } x = \cos x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}$$

$$\sin x = \cos x = \frac{-1}{\sqrt{2}} \Rightarrow x = \frac{5\pi}{4}$$

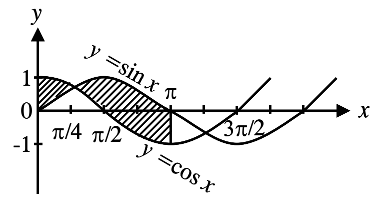


Fig. 7.19

From the figure we see that $\cos x > \sin x$ for $0 \le x < \frac{\pi}{4}$ and $\sin x > \cos x$ for $\frac{\pi}{4} < x < \pi$

$$\therefore \text{ Area } A = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx$$

$$= (\sin x + \cos x)_0^{\pi/4} + (-\cos x - \sin x)_{\pi/4}^{\pi}$$

$$= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) - (\sin 0 + \cos 0) + (-\cos\pi - \sin \pi) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4}\right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1) + (1 - 0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 2\sqrt{2} \text{ sq. units.}$$

Example 7.31: Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution : The curve is symmetric about both axes.

$\therefore$ Area of the ellipse = 4 × Area of the ellipse in the I quadrant.

$$I = 4 \int_0^a y dx$$

$$= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2}\, dx$$

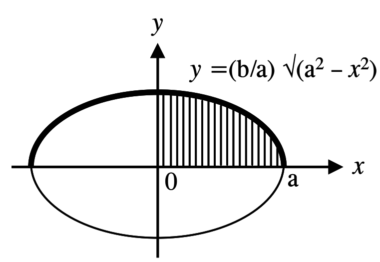


Fig. 7.20

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1) - 0\right] = \frac{4b}{a}\left(\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right)$$

$$= \pi\, ab \text{ sq. units.}$$

By using parametric form i.e., $4\int_0^a y\, dx = 4\int_0^{\pi/2} b \sin\theta\, (-\, a \sin\theta)\, d\theta$, we get the same area.

Example 7.32: Find the area of the curve $y^2 = (x-5)^2 (x-6)$

(i) between $x = 5$ and $x = 6$ (ii) between $x = 6$ and $x = 7$

Solution :

(i) $y^2 = (x-5)^2 (x-6)$

$\therefore y = (x-5)\sqrt{x-6}$

This curve cuts the x-axis at $x = 5$ and at $x = 6$

When x takes any value between 5 and 6, y^2 is negative.

$\therefore$ The curve does not exist in the interval $5 < x < 6$.

Hence the area between the curve at $x = 5$ and $x = 6$ is zero.

(ii) Required area $= \int_a^b y\,dx$

$$= 2\int_6^7 (x-5)\sqrt{x-6}\, dx$$

(Since the curve is symmetrical about x-axis)

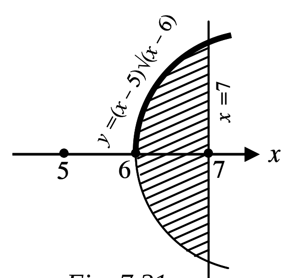


Fig. 7.21

$$= 2\int_6^7 (t+1)\sqrt{t}\, dt$$

$$= 2\int_0^1 (t^{3/2} + t^{1/2})dt$$

Take $t = x - 6$

$dt = dx$

$t = x - 6$

x	6	7
t	0	1

$$= 2\left[\frac{t^{5/2}}{\frac{5}{2}} + \frac{t^{3/2}}{\frac{3}{2}}\right]_0^1 = 2\left(\frac{2}{5} + \frac{2}{3}\right) = 2\left(\frac{6+10}{15}\right) = \frac{32}{15} \text{ sq. units}$$

Example 7.33: Find the area of the loop of the curve $3ay^2 = x(x-a)^2$

Solution :

Put $y = 0$; we get $x = 0, a$

It meets the x-axis at $x = 0$ and $x = a$

$\therefore$ Here a loop is formed between the points (0, 0) and (a, 0) about x-axis. Since the curve is symmetrical about x-axis, the area of the loop is twice the area of the portion above the x-axis.

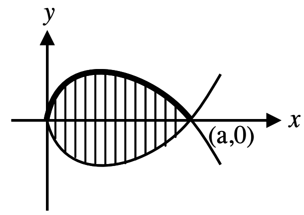


Fig. 7.22

$$\text{Required area} = 2\int_0^a y\,dx$$

$$= -2\int_0^a \frac{\sqrt{x}\,(x-a)}{\sqrt{3a}}\,dx = -\frac{2}{\sqrt{3a}}\int_0^a \left[x^{3/2} - a\sqrt{x}\right]dx$$

$$= -\frac{2}{\sqrt{3a}}\left[\frac{2}{5}x^{5/2} - \frac{2a}{3}x^{3/2}\right]_0^a = \frac{8a^2}{15\sqrt{3}} = \frac{8\sqrt{3}\,a^2}{45}$$

$$\therefore \text{ Required area } = \frac{8\sqrt{3}\,a^2}{45} \text{ sq. units.}$$

Example 7.34:

Find the area bounded by x-axis and an arch of the cycloid

$x = a\,(2t - \sin 2t)$, $y = a\,(1 - \cos 2t)$

Solution : The curves crosses x-axis when $y = 0$.

$\therefore\ a(1 - \cos 2t) = 0$

$\therefore\ \cos 2t = 1 \quad ; \quad 2t = 2n\pi,\ n \in z$

$\therefore\ t = 0, \pi, 2\pi, \dots$

$\therefore$ One arch of the curve lies between 0 and π

$$\text{Required area} = \int_a^b y\, dx$$

$$= \int_0^{\pi} a(1 - \cos 2t)\, 2a\,(1 - \cos 2t)\, dt \qquad \left| \begin{array}{l} y = a(1 - \cos 2t) \\ x = a\,(2t - \sin 2t) \\ dx = 2a(1 - \cos 2t)\, dt \end{array} \right.$$

$$= 2a^2 \int_0^{\pi} (1 - \cos 2t)^2 dt = 2a^2 \int_0^{\pi} (2 \sin^2 t)^2 dt = 8a^2 \int_0^{\pi} \sin^4 t\, dt$$

$$= 2 \times 8a^2 \int_0^{\pi/2} \sin^4 t\, dt \qquad \left(\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(2a - x) dx \right)$$

$$= 16a^2 \left[\frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \right] = 3\pi a^2 \text{ sq. units.}$$

7.5.2 Volume of solids of revolution :

Let f be a non-negative and continuous curve on $[a, b]$ and let R be the region bounded above by the graph of f, below by the x-axis and on the sides by the lines $x = a$ and $x = b$ *[Fig 6.23 (a)]*.

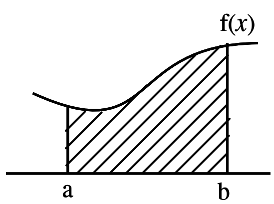


Fig. 7.23(a)

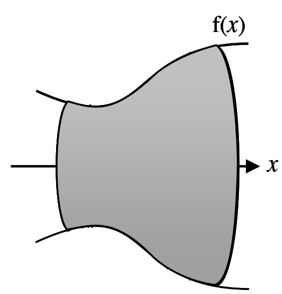


Fig. 7.23 (b)

When this region is revolved about the x-axis, it generates a solid having circular cross sections *(Fig. 7.23(b)]*. Since the cross section at x has radius $f(x)$, the cross-sectional area is $A(x) = \pi\, [f(x)]^2 = \pi y^2$

The volume of the solid is generated by moving the plane circular disc *[Fig.6.23(b)]* along x-axis perpendicular to the disc.

Therefore volume of the solid is $V = \int_a^b \pi\, [f(x)]^2 dx = \int_a^b \pi\, y^2\, dx$

(ii) If the region bounded by the graph of $x = g(y)$, the y-axis and on the sides by the lines $y = c$ and $y = d$ *(Fig. 7.24)* then the volume of the solid generated is given by

$$V = \int_c^d \pi\, [g(y)]^2 dx = \int_c^d \pi\, x^2\, dy$$

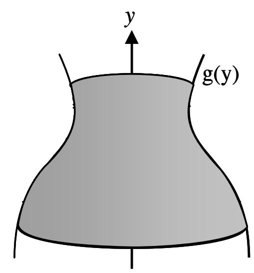


Fig. 7.24

Example 7.35:

Find the volume of the solid that results when the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$ is revolved about the minor axis.

Solution :

Volume of the solid is obtained by revolving the right side of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the y-axis.

Limits for y is obtained by putting $x = 0 \Rightarrow y^2 = b^2 \Rightarrow y = \pm b$

From the given curve $x^2 = \frac{a^2}{b^2}(b^2 - y^2)$

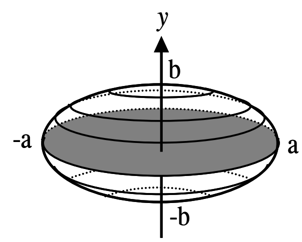


Fig. 7.25

$\therefore$ Volume is given by

$$V = \int_c^d \pi\, x^2 dy = \int_{-b}^{b} \pi\, \frac{a^2}{b^2}\, (b^2 - y^2)\, dy = 2\pi\, \frac{a^2}{b^2} \left(b^2 y - \frac{y^3}{3}\right)_0^b$$

$$= 2\pi\, \frac{a^2}{b^2}\left(b^3 - \frac{b^3}{3}\right) = \frac{4\pi}{3}\, a^2 b \text{ cubic units}$$

Example 7.36:

Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is revolved about the y-axis.

Solution : Since the solid is generated by revolving about the y-axis, rewrite $y = \sqrt{x}$ as $x = y^2$.

Taking the limits for y, $y = 0$ and $y = 2$ (putting $x = 0$ in $x = y^2$, we get $y = 0$)

Volume is given by $V = \int_c^d \pi x^2 dy$

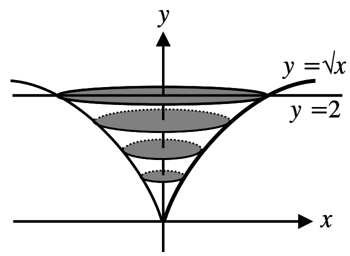


Fig. 7.26

$$= \int_0^2 \pi y^4 \, dy = \left[\frac{\pi y^5}{5}\right]_0^2 = \frac{32\pi}{5} \text{ cubic units.}$$

EXERCISE 7.4

(1) Find the area of the region bounded by the line $x - y = 1$ and
(i) x-axis, $x = 2$ and $x = 4$ (ii) x-axis, $x = -2$ and $x = 0$

(2) Find the area of the region bounded by the line $x - 2y - 12 = 0$ and
(i) y-axis, $y = 2$ and $y = 5$ (ii) y-axis, $y = -1$ and $y = -3$

(3) Find the area of the region bounded by the line $y = x - 5$ and the x-axis between the ordinates $x = 3$ and $x = 7$.

(4) Find the area of the region bounded by the curve $y = 3x^2 - x$ and the x-axis between $x = -1$ and $x = 1$.

(5) Find the area of the region bounded by $x^2 = 36y$, y-axis, $y = 2$ and $y = 4$.

(6) Find the area included between the parabola $y^2 = 4ax$ and its latus rectum.

(7) Find the area of the region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ between the two latus rectums.

(8) Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $2x - y = 4$.

(9) Find the common area enclosed by the parabolas $4y^2 = 9x$ and $3x^2 = 16y$

(10) Find the area of the circle whose radius is a

Find the volume of the solid that results when the region enclosed by the given curves : (11 to 14)

(11) $y = 1 + x^2$, $x = 1$, $x = 2$, $y = 0$ is revolved about the x-axis.

(12) $2ay^2 = x(x - a)^2$ is revolved about x-axis, $a > 0$.

(13) $y = x^3$, $x = 0$, $y = 1$ is revolved about the y-axis.

(14) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about major axis $a > b > 0$.

(15) Derive the formula for the volume of a right circular cone with radius '*r*' and height '*h*'.

(16) The area of the region bounded by the curve $xy = 1$, x-axis, $x = 1$. Find the volume of the solid generated by revolving the area mentioned about x-axis.

7.6. Length of the curve :

(i) If the function $f(x)$ and its derivative $f'(x)$ are continuous on $[a, b]$ then the arc length L of the curve $y = f(x)$ from $x = a$ to $x = b$ is defined to be $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

(ii) Similarly for a curve expressed in the form $x = g(y)$, where g is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ is given by $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

(iii) When the equation of the curve $y = f(x)$ is represented in parametric form $x = \phi(t)$, $y = \Psi(t)$, $\alpha \le t \le \beta$ where $\phi(t)$ and $\Psi(t)$ are continuous function with continuous derivatives and $\phi'(t)$ does not vanish in the given interval then $L = \int_\alpha^\beta \sqrt{(\phi'(t))^2 + (\Psi'(t))^2}\, dt$

7.7 Surface area of a solid :

(i) If the function $f(x)$ and its derivatives $f'(x)$ are continuous on $[a, b]$, then the surface area of the solid of revolution obtained by the revolution about x-axis, the area bounded by the curve $y = f(x)$ the two ordinates $x = a$, $x = b$ and x-axis is

$$S.A. = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

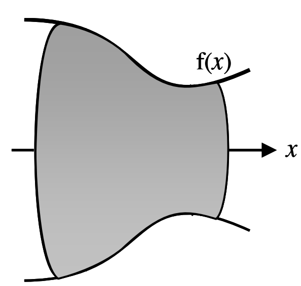


Fig. 7.27

(ii) Similarly for the curve expressed in the form $x = g(y)$ where $g'(y)$ is continuous on $[c, d]$, the surface area of the solid of revolution obtained by the revolution about y-axis, the area bounded by the curve $x = g(y)$ the two abscissa $y = c$, $y = d$ and y axis is

$$S.A. = 2\pi \int_c^d y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

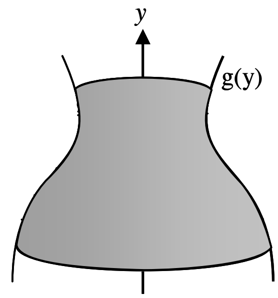


Fig. 7.28

(iii) When the equation of the curve $y = f(x)$ is represented in parametric form $x = g(t)$, $y = h(t)$, $\alpha \le t \le \beta$ where $g(t)$ and $h(t)$ are continuous function with continuous derivatives and $g'(t)$ does not vanish in the interval, then $S.A. = 2\pi \int_{t=\alpha}^{t=\beta} y \sqrt{(g'(t))^2 + (h'(t))^2}\, dt.$

Example 7.37: Find the length of the curve $4y^2 = x^3$ between $x = 0$ and $x = 1$

Solution :

$4y^2 = x^3$

Differentiating with respect to x

$$8y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{8y}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{9x^4}{64y^2}}$$

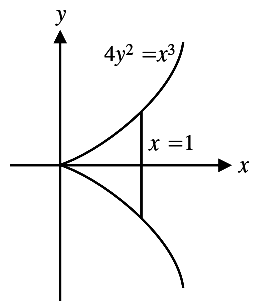


Fig. 7.29

$$= \sqrt{1 + \frac{9x^4}{16 \times 4y^2}} = \sqrt{1 + \frac{9x^4}{16x^3}} = \sqrt{1 + \frac{9x}{16}}$$

The curve is symmetrical about x-axis.

The required length

$$L = 2\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\int_0^1 \left(1 + \frac{9x}{16}\right)^{1/2} dx$$

$$= 2 \times \left[\frac{\left(1 + \frac{9x}{16}\right)^{3/2}}{\frac{9}{16} \times \frac{3}{2}} \right]_0^1 = \frac{64}{27}\left[\left(1 + \frac{9x}{16}\right)^{3/2}\right]_0^1$$

$$= \frac{64}{27}\left[\frac{125}{64} - 1\right] = \frac{61}{27}$$

Example 7.38: Find the length of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$

Solution :

$x = a\cos^3 t$, $y = a\sin^3 t$ is the parametric form of the given astroid, where $0 \le t \le 2\pi$

$$\frac{dx}{dt} = -3a\cos^2 t \sin t \;;$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

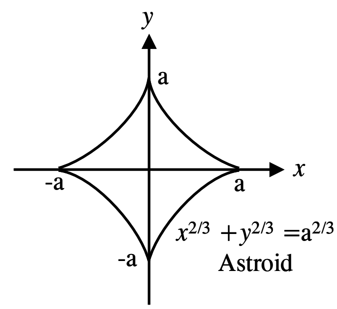


Fig. 7.30

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t} = 3a\sin t\cos t$$

Since the curve is symmetrical about both axes, the total length of the curve is 4 times the length in the first quadrant.

But t varies from 0 to $\frac{\pi}{2}$ in the first quadrant.

$$\therefore \text{ Length of the entire curve} = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 4\int_0^{\pi/2} 3a\sin t\cos t\, dt = 6a\int_0^{\pi/2} \sin 2t\, dt$$

$$= 6a \,.\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2} = -3a\,[\cos\pi - \cos 0]$$

$$= -3a\,[-1 - 1] = 6a$$

Example 7.39: Show that the surface area of the solid obtained by revolving the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about x-axis is $2\pi\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$

Solution : $y = \sin x$

Differentiating with respect to x $\dfrac{dy}{dx} = \cos x$.

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \cos^2 x}$$

$$\text{Surface area} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

when the area is rotated about the x-axis.

$$S = \int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x}\; dx$$

Put $\cos x = t$ $\qquad$ $t = \cos x$

$-\sin x\, dx = dt$

x	0	π
t	1	-1

$$= \int_1^{-1} 2\pi \sqrt{1 + t^2}\,(-dt) = 4\pi \int_0^1 \sqrt{1 + t^2}\,(dt)$$

$$= 4\pi\left[\frac{t}{2}\sqrt{1 + t^2} + \frac{1}{2}\log\left(t + \sqrt{1 + t^2}\right)\right]_0^1$$

$$= 2\pi\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right] - 0$$

$$= 2\pi\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$$

Example 7.40: Find the surface area of the solid generated by revolving the cycloid $x = a(t + \sin t)$, $y = a(1 + \cos t)$ about its base (x-axis).

Solution : $y = 0 \Rightarrow 1 + \cos t = 0 \quad \cos t = -1 \quad \Rightarrow \quad t = -\pi, \pi$

$$x = a\,(t + \sin t)\ ;\ y = a\,(1 + \cos t)$$

$$\frac{dx}{dt} = a\,(1 + \cos t) \quad \frac{dy}{dt} = -a \sin t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2\,(1 + \cos t)^2 + a^2 \sin^2 t} = 2a \cos\frac{t}{2}$$

$$\text{Surface area} = \int_{-\pi}^{\pi} 2\pi a\,(1 + \cos t)\, 2a \cos \frac{t}{2}\, dt$$

$$= \int_{-\pi}^{\pi} 2\pi\, a \,.\, 2 \cos^2 \frac{t}{2} \,.\, 2\, a \cos \frac{t}{2}\, dt \;= 16\pi\, a^2 \int_{0}^{\pi} \cos^3 \frac{t}{2}\, dt$$

$$= 16\pi a^2 \int_{0}^{\pi/2} 2\cos^3 x\, dx \quad \left[\text{Take } \frac{t}{2} = x\right]$$

$$= 32\pi a^2 I_3 = 32\pi a^2 \times \frac{2}{3}$$

$$= \frac{64}{3}\pi a^2 \text{ sq. units.}$$

EXERCISE 7.5

(1) Find the perimeter of the circle with radius a.

(2) Find the length of the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$ between $t = 0$ and π.

(3) Find the surface area of the solid generated by revolving the arc of the parabola $y^2 = 4ax$, bounded by its latus rectum about x-axis.

(4) Prove that the curved surface area of a sphere of radius r intercepted between two parallel planes at a distance a and b from the centre of the sphere is $2\pi r\,(b - a)$ and hence deduct the surface area of the sphere. $(b > a)$.

8. DIFFERENTIAL EQUATIONS

8.1. Introduction :

One of the branches of Mathematics conveyed clearly in the principal language of science called “Differential equations”, plays an important role in Science, Engineering and Social Sciences. Let us analyse a few of the examples cited below.

(1) Suppose that there are two living species which depend for their survival on a common source of food supply. This fact results in a competition in consuming the available food. The phenomenon, is commonly noticed in the plant life having common supply of water, fertilizer and minerals. However, whenever the competition between two species begins, the growth rate of one is retarded and we can note that the rate of retardation is naturally proportional to the size of the other species present at time t. This situation can be expressed as a Mathematical model whose solution would help us to determine the time at which one species would become extinct.

(2) Several diseases are caused by spread of an infection. Suppose that the susceptible population of a town is p. One person gets the infection. Because of contact another susceptible person is also infected. This process continues to cover the entire susceptible population. With some assumptions to simplify the mathematical considerations this situation can be framed into a mathematical model and a solution can be determined which would provide informations regarding the spread of the epidemic in the town.

(3) If a dead body is brought for a medical examination at a particular time, the exact time of death can be determined by noting the temperature of the body at various time intervals, formulating it into a mathematical problem with available initial conditions and then solving it.

(4) The determination of the amount of a radioactive material that disintegrates over a period of time is yet another mathematical formulation which yield the required result.

(5) Several examples exist in which two nations have disputes on various issues. Each nation builds its own arms to defend the nation from attack. Naturally a spirit of race in building up arms persists between conflicting nations. A small grievance quite often creates a war-like

situation and adds to increasing the level of arms. These commonly experienced facts can be presented in a mathematical language and hence solved. It is a fact that such a model has been tested for some realistic situations that had prevailed in the First and Second World War between conflicting nations.

From the above examples it is found that the mathematical formulation to all situations turn out to be differential equations. Thus the latent significance of differential equations in studying physical phenomena becomes apparent. This branch of Mathematics called 'Differential Equations' is like a bridge linking Mathematics and Science with its applications. Hence it is rightly considered as the language of Sciences.

Galileo once conjectured that the velocity of a body falling from rest is proportional to the distance fallen. Later he decided that it is proportional to the time instead. Each of these statements can be formulated as an equation involving the rate of change of an unknown function and is therefore an example of what Mathematicians call a Differential Equation. Thus $\frac{ds}{dt} = kt$ is a differential equation which gives velocity of a falling body from a distance s proportional to the time t.

Definition: An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a Differential Equation.

If $y = f(x)$ is a given function, then its derivative $\frac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x. In any natural process the variables involved and their rates of change are connected with one another by means of the basic scientific principles that govern the process. When this expression is written in mathematical symbols, the result is often a differential equation.

Thus a differential equation is an equation in which differential coefficients occur. Its importance can further be realised from the fact that every natural phenomena is governed by differential equations.

Differential equation are of two types.

(i) Ordinary and (ii) Partial.

In this chapter we concentrate only on Ordinary differential equations.

Definition : An ordinary differential equation is a differential equation in which a single independent variable enters either explicitly or implicitly.

For instance (i) $\frac{dy}{dx} = x + 5$ (ii) $(y')^2 + (y')^3 + 3y = x^2$ (iii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

are all ordinary differential equations.

8.2 Order and degree of a differential equation :

Definition : The **order** of a differential equation is the order of the highest order derivative occurring in it. The **degree** of the differential equation is the degree of the highest order derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

The degree of a differential equation does not require variables r, s, t … to be free from radicals and fractions.

Example 8.1: Find the order and degree of the following differential equations:

(i) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + y = 7$ (ii) $y = 4\frac{dy}{dx} + 3x\frac{dx}{dy}$

(iii) $\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$ (iv) $(1 + y')^2 = y'^2$

Solution : (i) The order of the highest derivative in this equation is 3. The degree of the highest order is 1. $\therefore$ (order, degree) = (3, 1)

(ii) $$y = 4\frac{dy}{dx} + 3x\frac{dx}{dy} \Rightarrow y = 4\left(\frac{dy}{dx}\right) + 3x\frac{1}{\left(\frac{dy}{dx}\right)}$$

Making the above equation free from fractions involving $\frac{dy}{dx}$ we get

$$y \cdot \frac{dy}{dx} = 4\left(\frac{dy}{dx}\right)^2 + 3x$$

Highest order = 1

Degree of Highest order = 2

(order, degree) = (1, 2)

(iii) $\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

To eliminate the radical in the above equation, raising to the power 4 on both sides, we get $\left(\frac{d^2y}{dx^2}\right)^4 = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^3$. Clearly (order, degree) = (2, 4).

(iv) $(1 + y')^2 = y'^2 \Rightarrow 1 + y'^2 + 2y' = y'^2$ from which it follows that

$$2\frac{dy}{dx} + 1 = 0 \quad \therefore \text{(order, degree)} = (1, 1).$$

8.3 Formation of differential equations :

Let $f(x, y, c_1) = 0$ be an equation containing x, y and one arbitrary constant c_1. If c_1 is eliminated by differentiating $f(x, y, c_1) = 0$ with respect to the independent variable once, we get a relation involving x, y and $\frac{dy}{dx}$, which is evidently a differential equation of the first order. Similarly, if we have an equation $f(x, y, c_1, c_2) = 0$ containing two arbitrary constants c_1 and c_2, then by differentiating this twice, we get three equations (including f). If the two arbitrary constants c_1 and c_2 are eliminated from these equations, we get a differential equation of second order.

In general if we have an equation $f(x, y, c_1, c_2, \ldots c_n) = 0$ containing n arbitrary constants $c_1, c_2 \ldots c_n$, then by differentiating n times we get $(n + 1)$ equations in total. If the n arbitrary constants $c_1, c_2, \ldots c_n$ are eliminated we get a differential equation of order n.

Note : If there are relations involving these arbitrary constants then the order of the differential equation may reduce to less than n.

Illustration :

Let us find the differential equation of straight lines $y = mx + c$ where both m and c are arbitrary constants.

Since m and c are two arbitrary constants differentiating twice we get

$$\frac{dy}{dx} = m$$

$$\frac{d^2y}{dx^2} = 0$$

Both the constants m and c are seen to be eliminated. Therefore the required differential equation is $\frac{d^2y}{dx^2} = 0$

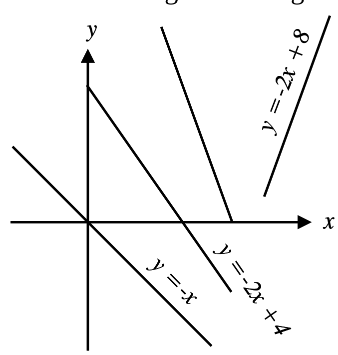


Fig. 8.1

Note : In the above illustration we have taken both the constants m and c as arbitrary. Now the following two cases may arise.

Case (i) : m is arbitrary and c is fixed. Since m is the only arbitrary constant in $y = mx + c$; … (1)

Differentiating once we get

$$\frac{dy}{dx} = m \quad \dots (2)$$

Eliminating m between (1) and (2) we get the required differential equation

$$x\left(\frac{dy}{dx}\right) - y + c = 0$$

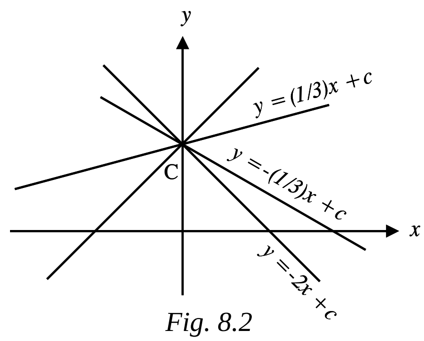


Fig. 8.2

Case (ii) : c is an arbitrary constant and m is a fixed constant.

Since c is the only arbitrary constant differentiating once we get $\frac{dy}{dx} = m$. Clearly c is eliminated from the above equation. Therefore the required differential equation is $\frac{dy}{dx} = m$.

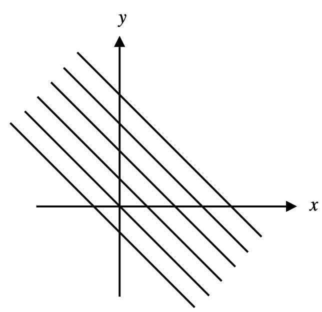


Fig. 8.3

Example 8.2: Form the differential equation from the following equations.

(i) $y = e^{2x}(A + Bx)$ (ii) $y = e^{x}(A\cos 3x + B\sin 3x)$

(iii) $Ax^2 + By^2 = 1$ (iv) $y^2 = 4a(x - a)$

Solution :

(i) $y = e^{2x}(A + Bx)$

$$ye^{-2x} = A + Bx \quad \dots (1)$$

Since the above equation contains two arbitrary constants, differentiating twice, we get $y'e^{-2x} - 2y\,e^{-2x} = B$

$$\{y''e^{-2x} - 2y'\,e^{-2x}\} - 2\{y'e^{-2x} - 2y\,e^{-2x}\} = 0$$

$$e^{-2x}\{y'' - 4y' + 4y\} = 0 \qquad [\because e^{-2x} \neq 0]$$

$y'' - 4y' + 4y = 0$ is the required differential equation.

(ii) $y = e^x (A \cos 3x + B \sin 3x)$

$ye^{-x} = A \cos 3x + B \sin 3x$

We have to differentiate twice to eliminate two arbitrary constants

$$y'e^{-x} - ye^{-x} = -3A \sin 3x + 3B \cos 3x$$

$$y'' e^{-x} - y'e^{-x} - y'e^{-x} + ye^{-x} = -9(A \cos 3x + B \sin 3x)$$

$$\text{i.e., } e^{-x}(y'' - 2y' + y) = -9ye^{-x}$$

$$\Rightarrow y'' - 2y' + 10y = 0 \quad (\because e^{-x} \neq 0)$$

(iii) $Ax^2 + By^2 = 1$ … (1)

Differentiating, $2Ax + 2Byy' = 0$ i.e., $Ax + Byy' = 0$ … (2)

Differentiating again, $A + B(yy'' + y'^2) = 0$ … (3)

Eliminating A and B between (1), (2) and (3) we get

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & yy' & 0 \\ 1 & yy'' + y'^2 & 0 \end{vmatrix} = 0 \Rightarrow (yy'' + y'^2)x - yy' = 0$$

(iv) $y^2 = 4a(x - a)$ … (1)

Differentiating, $2yy' = 4a$ … (2)

Eliminating a between (1) and (2) we get

$$y^2 = 2yy'\left(x - \frac{yy'}{2}\right)$$

$$\Rightarrow (yy')^2 - 2xyy' + y^2 = 0$$

EXERCISE 8.1

(1) Find the order and degree of the following differential equations.

(i) $\dfrac{dy}{dx} + y = x^2$

(ii) $y' + y^2 = x$

(iii) $y'' + 3y'^2 + y^3 = 0$

(iv) $\dfrac{d^2y}{dx^2} + x = \sqrt{y + \dfrac{dy}{dx}}$

(v) $\dfrac{d^2y}{dx^2} - y + \left(\dfrac{dy}{dx} + \dfrac{d^3y}{dx^3}\right)^{\frac{3}{2}} = 0$

(vi) $y'' = (y - y'^3)^{\frac{2}{3}}$

(vii) $y' + (y'')^2 = (x + y'')^2$

(viii) $y' + (y'')^2 = x(x + y'')^2$

(ix) $\left(\dfrac{dy}{dx}\right)^2 + x = \dfrac{dx}{dy} + x^2$

(x) $\sin x\,(dx + dy) = \cos x\,(dx - dy)$

(2) Form the differential equations by eliminating arbitrary constants given in brackets against each

(i)	$y^2 = 4ax$	$\{a\}$
(ii)	$y = ax^2 + bx + c$	$\{a, b\}$
(iii)	$xy = c^2$	$\{c\}$
(iv)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\{a, b\}$
(v)	$y = Ae^{2x} + Be^{-5x}$	$\{A, B\}$
(vi)	$y = (A + Bx)e^{3x}$	$\{A, B\}$
(vii)	$y = e^{3x}\{C\cos 2x + D\sin 2x)$	$\{C, D\}$
(viii)	$y = e^{mx}$	$\{m\}$
(ix)	$y = Ae^{2x}\cos(3x + B)$	$\{A, B\}$

(3) Find the differential equation of the family of straight lines $y = mx + \frac{a}{m}$ when (i) m is the parameter ; (ii) a is the parameter ; (iii) a, m both are parameters

(4) Find the differential equation that will represent the family of all circles having centres on the x-axis and the radius is unity.

8.4 Differential equations of first order and first degree :

In this section we consider a class of differential equations, the order and degree of each member of the class is equal to one. For example,

(i) $yy' + x = 0$ (ii) $y' + xy = \sin x$ (iii) $y' = \frac{x+y}{x-y}$ (iv) $x\,dy + y\,dx = 0$

Solutions of first order and first degree equations:

We shall consider only certain special types of equations of the first order and first degree. viz., (i) Variable separable (ii) Homogeneous (iii) Linear.

8.4.1 Variable separable :

Variables of a differential equation are to be rearranged in the form

$f_1(x)\,g_2(y)\,dx + f_2(x)\,g_1(y)\,dy = 0$

i.e., the equation can be written as

$$f_2(x)g_1(y)dy = -f_1(x)\,g_2(y)\,dx$$

$$\Rightarrow \frac{g_1(y)}{g_2(y)}\,dy = -\frac{f_1(x)}{f_2(x)}\,dx$$

The solution is therefore given by $\int \frac{g_1(y)}{g_2(y)}\,dy = -\int \frac{f_1(x)}{f_2(x)}\,dx + c$

Example 8.3: Solve : $\frac{dy}{dx} = 1 + x + y + xy$

Solution : The given equation can be written in the form

$$\frac{dy}{dx} = (1 + x) + y(1 + x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x)\,(1 + y)$$

$$\Rightarrow \quad \frac{dy}{1 + y} = (1 + x)dx$$

Integrating, we have

$\log(1 + y) = x + \frac{x^2}{2} + c$, which is the required solution.

Example 8.4: Solve $3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$

Solution : The given equation can be written in the form

$$\frac{3e^x}{1 + e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integrating, we have

$$3 \log(1 + e^x) + \log \tan y = \log c$$

$$\Rightarrow \quad \log[\tan y\,(1 + e^x)^3] = \log c$$

$\Rightarrow \quad (1 + e^x)^3 \tan y = c$, which is the required solution.

Note : The arbitrary constant may be chosen like c, $\frac{1}{c}$, $\log c$, e^c etc depending upon the problem.

Example 8.5: Solve $\frac{dy}{dx} + \left(\frac{1 - y^2}{1 - x^2}\right)^{\frac{1}{2}} = 0$

Solution : The given equation can be written as

$$\frac{dy}{dx} = -\left(\frac{1 - y^2}{1 - x^2}\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dy}{\sqrt{1 - y^2}} = -\frac{dx}{\sqrt{1 - x^2}}$$

Integrating, we have $\sin^{-1} y + \sin^{-1} x = c$

$\Rightarrow \quad \sin^{-1}\left[x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right] = c$

$\Rightarrow \quad x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = C$ is the required solution.

Example 8.6: Solve : $e^x \sqrt{1-y^2}\, dx + \frac{y}{x} dy = 0$

Solution : The given equation can be written as

$$xe^x dx = \frac{-y}{\sqrt{1-y^2}} dy$$

Integrating, we have

$$\int xe^x \, dx = -\int \frac{y}{\sqrt{1-y^2}} dy$$

$$\Rightarrow \; xe^x - \int e^x \, dx = \frac{1}{2} \int \frac{dt}{\sqrt{t}} \text{ where } t = 1 - y^2 \text{ so that } -2y\, dy = dt$$

$$\Rightarrow \; xe^x - e^x = \frac{1}{2}\left(\frac{t^{\frac{1}{2}}}{1/2}\right) + c$$

$$\Rightarrow xe^x - e^x = \sqrt{t} + c$$

$$\Rightarrow xe^x - e^x - \sqrt{1-y^2} = c \quad \text{which is the required solution.}$$

Example 8.7: Solve : $(x + y)^2 \frac{dy}{dx} = a^2$

Solution : Put $x + y = z$. Differentiating with respect to x we get

$$1 + \frac{dy}{dx} = \frac{dz}{dx} \quad \text{i.e.,} \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$$

The given equation becomes $z^2\left(\frac{dz}{dx} - 1\right) = a^2$

$$\Rightarrow \; \frac{dz}{dx} - 1 = \frac{a^2}{z^2} \quad \text{or} \quad \frac{z^2}{z^2 + a^2} dz = dx$$

Integrating we have, $\displaystyle\int \frac{z^2}{z^2 + a^2} \, dz = \int dx$

$$\int \frac{z^2 + a^2 - a^2}{z^2 + a^2} dz = x + c \Rightarrow \int \left(1 - \frac{a^2}{z^2 + a^2}\right) dz = x + c$$

$$\Rightarrow \; z - a^2 \cdot \frac{1}{a} \tan^{-1} \frac{z}{a} = x + c$$

$$\Rightarrow \; x + y - a \tan^{-1}\left(\frac{x + y}{a}\right) = x + c \qquad (\because z = x + y)$$

$$\text{i.e., } \; y - a \tan^{-1}\left(\frac{x + y}{a}\right) = c, \; \text{ which is the required solution.}$$

Example 8.8: Solve : $x\,dy = (y + 4x^5 e^{x^4})dx$

Solution :

$$xdy - y\,dx = 4x^5 e^{x^4} dx$$

$$\frac{xdy - ydx}{x^2} = 4x^3 e^{x^4} dx$$

Integrating we have, $\displaystyle\int \frac{xdy - ydx}{x^2} = \int 4x^3 e^{x^4} dx$

$$\Rightarrow \int d\left(\frac{y}{x}\right) = \int e^t\,dt \qquad \text{where } t = x^4$$

$$\Rightarrow \frac{y}{x} = e^t + c$$

i.e., $\dfrac{y}{x} = e^{x^4} + c$ which is the required solution.

Example 8.9: Solve: $(x^2 - y)dx + (y^2 - x)\,dy = 0$, if it passes through the origin.

Solution :

$$(x^2 - y)dx + (y^2 - x)\,dy = 0$$

$$x^2dx + y^2dy = xdy + ydx$$

$$x^2dx + y^2\,dy = d(xy)$$

Integrating we have, $\dfrac{x^3}{3} + \dfrac{y^3}{3} = xy + c$

Since it passes through the origin, $c = 0$

$\therefore$ the required solution is $\dfrac{x^3}{3} + \dfrac{y^3}{3} = xy$ or $x^3 + y^3 = 3xy$

Example 8.10 : Find the cubic polynomial in x which attains its maximum value 4 and minimum value 0 at $x = -1$ and 1 respectively.

Solution : Let the cubic polynomial be $y = f(x)$. Since it attains a maximum at $x = -1$ and a minimum at $x = 1$.

$$\frac{dy}{dx} = 0 \quad \text{at } x = -1 \text{ and } 1$$

$$\frac{dy}{dx} = k\,(x + 1)\,(x - 1) = k(x^2 - 1)$$

Separating the variables we have $dy = k(x^2 - 1)\,dx$

$$\int dy = k \int (x^2 - 1)\, dx$$

$$y = k\left(\frac{x^3}{3} - x\right) + c \qquad \ldots (1)$$

when $x = -1$, $y = 4$ and when $x = 1$, $y = 0$

Substituting these in equation (1) we have

$$2k + 3c = 12 \quad ; \quad -2k + 3c = 0$$

On solving we have $k = 3$ and $c = 2$. Substituting these values in (1) we get the required cubic polynomial $y = x^3 - 3x + 2$.

Example 8.11 : The normal lines to a given curve at each point (x, y) on the curve pass through the point (2, 0). The curve passes through the point (2, 3). Formulate the differential equation representing the problem and hence find the equation of the curve.

Solution :

Slope of the normal at any point $P(x, y) = -\frac{dx}{dy}$

Slope of the normal $AP = \frac{y - 0}{x - 2}$ $\quad \therefore -\frac{dx}{dy} = \frac{y}{x-2} \Rightarrow ydy = (2 - x)dx$

Integrating both sides, $\frac{y^2}{2} = 2x - \frac{x^2}{2} + c \qquad \ldots (1)$

Since the curve passes through (2, 3)

$$\frac{9}{2} = 4 - \frac{4}{2} + c \Rightarrow c = \frac{5}{2}\text{; put } c = \frac{5}{2} \text{ in (1),}$$

$$\frac{y^2}{2} = 2x - \frac{x^2}{2} + \frac{5}{2} \Rightarrow y^2 = 4x - x^2 + 5$$

EXERCISE 8.2

Solve the following :

(1) $\sec 2x\, dy - \sin 5x \sec^2 y dx = 0$

(2) $\cos^2 x dy + y e^{\tan x} dx = 0$

(3) $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

(4) $yx^2 dx + e^{-x} dy = 0$

(5) $(x^2 + 5x + 7)\, dy + \sqrt{9 + 8y - y^2}\, dx = 0$

(6) $\frac{dy}{dx} = \sin(x + y)$

(7) $(x + y)^2 \frac{dy}{dx} = 1$

(8) $ydx + xdy = e^{-xy}\, dx$ if it cuts the y-axis.

8.4.2 Homogeneous equations :

Definition :

A differential equation of first order and first degree is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ or $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$

Working rule for solving homogeneous equation :

By definition the given equation can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \ldots (1)$$

To solve (1) put $\quad y = \nu x \qquad \ldots (2)$

Differentiating (2) with respect to x gives

$$\frac{dy}{dx} = \nu + x\frac{d\nu}{dx} \qquad \ldots (3)$$

Using (2) and (3) in (1) we have

$$\nu + x\frac{d\nu}{dx} = f(\nu) \quad \text{or} \quad x\frac{d\nu}{dx} = f(\nu) - \nu$$

Seperating the variables x and ν we have

$$\frac{dx}{x} = \frac{d\nu}{f(\nu) - \nu} \Rightarrow \log x + c = \int \frac{d\nu}{f(\nu) - \nu}$$

where c is an arbitrary constant. After integration, replace ν by $\frac{y}{x}$.

Example 8.12: Solve : $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Solution : Put $y = vx$

$$\text{L.H.S.} = \nu + x\frac{d\nu}{dx} \ ; \ \text{R.H.S.} = v + \tan v$$

$$\therefore \ \nu + x\frac{d\nu}{dx} = \nu + \tan\nu \quad \text{or} \quad \frac{dx}{x} = \frac{\cos\nu}{\sin\nu}\, dv$$

Integrating, we have $\quad \log x = \log\sin\nu + \log c \Rightarrow x = c\sin\nu$

$$\text{i.e., } x = c\sin\left(\frac{y}{x}\right),$$

Example 8.13: Solve : $\left(2\sqrt{xy} - x\right) dy + ydx = 0$

Solution : The given equation is $\frac{dy}{dx} = \frac{-y}{2\sqrt{xy} - x}$

Put $\quad y = vx$

$$\text{L.H.S.} = v + x\frac{dv}{dx} \;;\; \text{R.H.S.} = \frac{-v}{2\sqrt{v}-1} = \frac{v}{1-2\sqrt{v}}$$

$$\therefore\; v + x\frac{dv}{dx} = \frac{v}{1-2\sqrt{v}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2\,v\sqrt{v}}{1-2\sqrt{v}} \Rightarrow \left(\frac{1-2\sqrt{v}}{v\sqrt{v}}\right)dv = 2\,\frac{dx}{x}$$

$$\text{i.e., } \left(v^{-3/2} - 2.\frac{1}{v}\right)dv = 2\,\frac{dx}{x}$$

$$\Rightarrow\; -2v^{-1/2} - 2\log v = 2\log x + 2\log c$$

$$-v^{-1/2} = \log(v\,x\,c)$$

$$-\sqrt{\frac{x}{y}} = \log(cy) \;\Rightarrow\; cy = e^{-\sqrt{x/y}} \text{ or } ye^{\sqrt{x/y}} = c$$

Note : This problem can also be done easily by taking $x = vy$

Example 8.14: Solve : $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$

Solution : $$\frac{dy}{dx} = -\frac{x^3+3xy^2}{y^3+3x^2y}$$

Put $y = \nu x$

$$\text{L.H.S.} = \nu + x\frac{d\nu}{dx} \;;\; \text{R.H.S.} = -\frac{x^3+3xy^2}{y^3+3x^2y} = -\left(\frac{1+3\nu^2}{\nu^3+3\nu}\right)$$

$$\therefore\; \nu + x\frac{d\nu}{dx} = -\left(\frac{1+3\nu^2}{\nu^3+3\nu}\right)$$

$$\Rightarrow\; x\frac{d\nu}{dx} = -\frac{\nu^4+6\nu^2+1}{\nu^3+3\nu}$$

$$\Rightarrow\; \frac{4dx}{x} = -\frac{4\nu^3+12\nu}{\nu^4+6\nu^2+1}\,d\nu$$

Integrating, we have

$$4\log x = -\log(\nu^4+6\nu^2+1) + \log c$$

$$\log[x^4(\nu^4+6\nu^2+1)] = \log c$$

$$\text{i.e., } x^4(\nu^4+6\nu^2+1) = c \quad \text{or}$$

$$y^4 + 6x^2y^2 + x^4 = c$$

Note (i) : This problem can also be done by using variable separable method.

Note (ii) : Sometimes it becomes easier in solving problems of the type $\frac{dx}{dy} = \frac{f_1(x/y)}{f_2(x/y)}$. The following example explains this case.

Example 8.15:

Solve : $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)\, dy = 0$ given that $y = 1$, where $x = 0$

Solution : The given equation can be written as

$$\frac{dx}{dy} = \frac{(x / y - 1)e^{x/y}}{1 + e^{x/y}} \qquad \ldots (1)$$

Put $x = \nu y$

$$\text{L.H.S.} = \nu + y\frac{d\nu}{dy} \; ; \; \text{R.H.S.} = \frac{(\nu - 1)e^{\nu}}{1 + e^{\nu}}$$

$$\therefore \; \nu + y\frac{d\nu}{dy} = \frac{(\nu - 1)e^{\nu}}{1 + e^{\nu}}$$

$$\text{or } y\frac{d\nu}{dy} = -\frac{(e^{\nu} + \nu)}{1 + e^{\nu}}$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{(e^{\nu} + 1)}{e^{\nu} + \nu}\, d\nu$$

Integrating we have, $\quad \log y = -\log(e^{\nu} + \nu) + \log c$

or $y(e^{\nu} + \nu) = c \;\; \Rightarrow ye^{x/y} + x = c$

Now $y = 1$ when $x = 0 \Rightarrow \; 1e^0 + 0 = c \Rightarrow c = 1$

$\therefore \; ye^{x/y} + x = 1$

Example 8.16: Solve : $xdy - ydx = \sqrt{x^2 + y^2}\, dx$

Solution : From the given equation we have

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \qquad \ldots (1)$$

Put $y = \nu x$

$$\text{L.H.S.} = \nu + x\frac{d\nu}{dx} \; ; \; \text{R.H.S.} = \frac{\nu + \sqrt{1 + \nu^2}}{1}$$

$$\therefore \; \nu + x\frac{d\nu}{dx} = \nu + \sqrt{1 + \nu^2} \quad \text{or} \quad \frac{dx}{x} = \frac{d\nu}{\sqrt{1 + \nu^2}}$$

Integrating, we have, $\log x + \log c = \log\left[v + \sqrt{v^2 + 1}\right]$

$$\text{i.e.,}\quad xc = v + \sqrt{v^2 + 1} \Rightarrow x^2 c = y + \sqrt{(y^2 + x^2)}$$

EXERCISE 8.3

Solve the following :

(1) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ (2) $\frac{dy}{dx} = \frac{y(x-2y)}{x(x-3y)}$ (3) $(x^2 + y^2)\, dy = xy\, dx$

(4) $x^2 \frac{dy}{dx} = y^2 + 2xy$ given that $y = 1$, when $x = 1$.

(5) $(x^2 + y^2)\, dx + 3xy\, dy = 0$

(6) Find the equation of the curve passing through (1, 0) and which has slope $1 + \frac{y}{x}$ at (x, y).

8.4.3 Linear Differential Equation :

Definition :

A first order differential equation is said to be linear in y if the power of the terms $\frac{dy}{dx}$ and y are unity.

For **example** $\frac{dy}{dx} + xy = e^x$ is linear in y, since the power of $\frac{dy}{dx}$ is one and also the power of y is one. If a term occurs in the form $y\frac{dy}{dx}$ or y^2, then it is not linear, as the degree of each term is two.

A differential equation of order one satisfying the above condition can always be put in the form $\frac{dy}{dx} + Py = Q$, where P and Q are function of x only. Similarly a first order linear differential equation in x will be of the form $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y only.

The solution of the equation which is linear in y is given as

$y e^{\int P dx} = \int Q e^{\int P dx}\, dx + c$ where $e^{\int P dx}$ is known as an integrating factor and it is denoted by I.F.

Similarly if an equation is linear in x then the solution of such an equation becomes

$$x\,e^{\int P dy} = \int Q\, e^{\int P dy}\, dy + c \qquad \left(\text{where } e^{\int P dy} \text{ is I.F.}\right)$$

We frequently use the following properties of logarithmic and exponential functions :

(i) $e^{\log A} = A$ (ii) $e^{m \log A} = A^m$ (iii) $e^{-m \log A} = \dfrac{1}{A^m}$

Example 8.17 : Solve : $\dfrac{dy}{dx} + y \cot x = 2 \cos x$

Solution : The given equation is of the form $\dfrac{dy}{dx} + Py = Q$. This is linear in y.

Here $P = \cot x$ and $Q = 2 \cos x$

I.F. $= e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

$\therefore$ The required solution is

$$y \text{ (I.F.)} = \int (Q \text{ (I.F.)})\, dx + c \Rightarrow y(\sin x) = \int 2 \cos x \sin x\, dx + c$$

$$\Rightarrow y \sin x = \int \sin 2x\, dx + c$$

$$\Rightarrow y \sin x = -\frac{\cos 2x}{2} + c$$

$$\Rightarrow 2y \sin x + \cos 2x = c$$

Example 8.18 : Solve : $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{(1 - x^2)}$

Solution: The given equation is $\dfrac{dy}{dx} + \left(\dfrac{2x}{1 - x^2}\right) y = \dfrac{x}{\sqrt{(1 - x^2)}}$. This is linear in y

$$\text{Here } \int P dx = \int \frac{2x}{1 - x^2}\, dx = -\log(1 - x^2)$$

$$\text{I.F.} = e^{\int P dx} = \frac{1}{1 - x^2}$$

The required solution is

$$y \cdot \frac{1}{1 - x^2} = \int \frac{x}{\sqrt{(1 - x^2)}} \times \frac{1}{1 - x^2}\, dx. \quad \text{Put } 1 - x^2 = t \Rightarrow -2x dx = dt$$

$$\therefore \frac{y}{1 - x^2} = \frac{-1}{2} \int t^{-3/2}\, dt + c$$

$$\Rightarrow \quad \frac{y}{1-x^2} = t^{-1/2} + c$$

$$\Rightarrow \quad \frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c$$

Example 8.19 : Solve : $(1 + y^2)dx = (\tan^{-1}y - x)dy$

Solution : The given equation can be written as $\dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1}y}{1+y^2}$.

This is linear in x. Therefore we have

$$\int P dy = \int \frac{1}{1+y^2}\, dy = \tan^{-1}y$$

$$\text{I.F.} = e^{\int P dy} = e^{\tan^{-1}y}$$

The required solution is

$$x e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \left(\frac{\tan^{-1}y}{1+y^2}\right) dy + c \qquad \left\{\begin{array}{l}\text{put } \tan^{-1}y = t \\ \therefore \dfrac{dy}{1+y^2} = dt\end{array}\right.$$

$$\Rightarrow \quad x e^{\tan^{-1}y} = \int e^t \,.\, t \;\, dt + c$$

$$\Rightarrow \quad x e^{\tan^{-1}y} = te^t - e^t + c$$

$$\Rightarrow \quad x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

Example 8.20 : Solve : $(x + 1)\dfrac{dy}{dx} - y = e^x(x + 1)^2$

Solution : The given equation can be written as $\dfrac{dy}{dx} - \dfrac{y}{x+1} = e^x(x + 1)$

This is linear in y. Here $\int P dx = -\int \dfrac{1}{x+1}\, dx = -\log(x + 1)$

$$\text{So} \quad \text{I.F.} = e^{\int P dx} = e^{-\log(x+1)} = \frac{1}{x+1}$$

$\therefore$ The required solution is

$$y \cdot \frac{1}{x+1} = \int e^x (x+1) \frac{1}{x+1}\; dx + c$$

$$= \int e^x \, dx + c$$

$$\text{i.e.,} \quad \frac{y}{x+1} = e^x + c$$

Example 8.21 : Solve : $\frac{dy}{dx} + 2y \tan x = \sin x$

Solution : This is linear in y. Here $\int P dx = \int 2 \tan x \, dx = 2 \log \sec x$

$$\text{I.F.} = e^{\int P dx} = e^{\log \sec^2 x} = \sec^2 x$$

The required solution is

$$y \sec^2 x = \int \sec^2 x \,.\, \sin x \, dx$$
$$= \int \tan x \sec x \, dx$$
$$\Rightarrow y \sec^2 x = \sec x + c \quad \text{or} \quad y = \cos x + c \cos^2 x$$

EXERCISE 8.4

Solve the following :

(1) $\frac{dy}{dx} + y = x$

(2) $\frac{dy}{dx} + \frac{4x}{x^2 + 1} y = \frac{1}{(x^2 + 1)^2}$

(3) $\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$

(4) $(1 + x^2) \frac{dy}{dx} + 2xy = \cos x$

(5) $\frac{dy}{dx} + \frac{y}{x} = \sin(x^2)$

(6) $\frac{dy}{dx} + xy = x$

(7) $dx + x dy = e^{-y} \sec^2 y \, dy$

(8) $(y - x) \frac{dy}{dx} = a^2$

(9) Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y = 2(e^x - x - 1)$

8.5 Second order linear differential equations with constant coefficients :

A general second order non-homogeneous linear differential equation with constant coefficients is of the form

$a_0 y'' + a_1 y' + a_2 y = X$ … (1),

where a_0, a_1, a_2 are constants $a_0 \neq 0$, and X is a function of x. The equation

$a_0 y'' + a_1 y' + a_2 y = 0, \; a_0 \neq 0$ … (2)

is known as a homogeneous linear second order differential equation with constant coefficients,

To solve (1), first we solve (2). To do this we proceed as follows :

Consider the function $y = e^{px}$, p is a constant.

$$\text{Now } y' = p e^{px} \text{ and } y'' = p^2 e^{px}$$

Note that the derivatives look similar to the function $y = e^{px}$ itself and if $L(y) = a_0y'' + a_1y' + a_2y$ then

$$\begin{aligned} L(y) &= L(e^{px}) \\ &= (a_0p^2e^{px} + a_1pe^{px} + a_2\, e^{px}) \\ &= (a_0p^2 + a_1p + a_2)e^{px} \end{aligned}$$

Hence if $L(y) = 0$ then it follows that $(a_0p^2 + a_1p + a_2)e^{px} = 0$.

Since $e^{px} \neq 0$ we get that $a_0p^2 + a_1p + a_2 = 0$ … (3)

Note that e^{px} satisfies the equation $L(y) = a_0y'' + a_1y' + a_2y = 0$ then p must satisfy $a_0p^2 + a_1p + a_2 = 0$. Moreover if the various derivatives of a function look similar in form to the function itself then e^{px} will be an ideal candidate to solve $a_0y'' + a_1y' + a_2y = 0$. Hereafter we will consider only those set of differential equations which admits e^{px} as one of the solutions. Hence we have the following :

Theorem : If λ is a root of $a_0p^2 + a_1p + a_2 = 0$, then $e^{\lambda x}$ is a solution of $a_0y'' + a_1y' + a_2y = 0$

8.5.1 Definition : The equation $a_0p^2 + a_1p + a_2 = 0$ is called the characteristic equation of (2).

In general the characteristic equation has two roots say λ_1 and λ_2. Then the following three cases do arise.

Case (i) : λ_1 and λ_2 are real and distinct.

In this case, by the above theorem $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are solutions of (2), and the linear combination $y = c_1\, e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ is also a solution of (2).

For $L(y) = a_0(c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x})'' + a_1(c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x})' + a_2(c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x})$

$= c_1(a_0\lambda_1^2 + a_1\lambda_1 + a_2)e^{\lambda_1 x} + c_2(a_0\lambda_2^2 + a_1\,\lambda_2 + a_2)e^{\lambda_2 x} = c_1\,.\,0 + c_2\,.\,0 = 0.$

and the solution $c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ is known as the complementary function.

Case (ii) : λ_1 and λ_2 are complex $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$

In this case as the two roots λ_1 and λ_2 are complex from theory of equations

$e^{\lambda_1 x} = e^{(a + ib)x} = e^{ax}\,.\,e^{ibx} = e^{ax}(\cos bx + i\sin bx)$ and

$e^{\lambda_2 x} = e^{ax}$ (cos $bx - i$ sin bx)

Hence the solution

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = e^{ax}\left[(c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx\right]$$

$$= e^{ax}[A\cos bx + B\sin bx] \text{ where } A = c_1 + c_2 \text{ and } B = (c_1 - c_2)i$$

and the complementary function is e^{ax} [A cos bx + B sin bx].

Case (iii) : The roots are real and equal $\lambda_1 = \lambda_2$ (say)

Clearly $e^{\lambda_1 x}$ is one of the solutions of (2). By using the double root property, we will obtain $xe^{\lambda_1 x}$ as the other solution of (2). Now the linear combination $c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$ becomes the solution. i.e., $y = (c_1 + c_2 x)e^{\lambda_1 x}$ is the solution or C.F.

The above discussion is summarised as follows :

Given $a_0 y'' + a_1 y' + a_2 y = 0$

Determine its characteristic equation

$a_0 p^2 + a_1 p + a_2 = 0$ … (3).

Let λ_1, λ_2 be the two roots of (3), then the solution of (2) is

$$y = \begin{cases} Ae^{\lambda_1 x} + Be^{\lambda_2 x} \text{ if } \lambda_1 \text{ and } \lambda_2 \text{ are real and distinct} \\ e^{ax}(A\cos bx + B\sin bx) \text{ if } \lambda_1 = a + ib \text{ and } \lambda_2 = a - ib \\ (A + Bx)e^{\lambda_1 x} \text{ if } \lambda_1 = \lambda_2 \text{ (real)} \end{cases}$$

A and B are arbitrary constants.

General solution :

The general solution of a linear equation of second order with constant co-efficient consists of two parts namely the complementary function and the particular integral.

Working rule :

To obtain the complementary function (C.F.) we solve the equation

$a_0 \dfrac{d^2y}{dx^2} + a_1 \dfrac{dy}{dx} + a_2 y = 0$ and obtain a solution $y = u$ (say). Then the general solution is given by $y = u + v$ where v is called the particular integral of (1).

The function u, the complementary function is associated with the homogeneous equation and v, the particular integral is associated with the term X. If $X = 0$ then the $C.F.$ becomes the general solution of the equation.

Note : In this section we use the differential operators

$$D \equiv \frac{d}{dx} \text{ and } D^2 \equiv \frac{d^2}{dx^2} \quad ; \quad Dy = \frac{dy}{dx} \quad ; \quad D^2y = \frac{d^2y}{dx^2}$$

8.5.2 Method for finding Particular Integral :

(a) Suppose X is of the form $e^{\alpha x}$, α *a* constant

$$D(e^{\alpha x}) = \alpha e^{\alpha x} \quad ; \; D^2 (e^{\alpha x}) = \alpha^2 e^{\alpha x} \; \ldots$$

$$D^n(e^{\alpha x}) = \alpha^n e^{\alpha x}, \text{ then } f(D)\, e^{\alpha x} = f(\alpha)\, e^{\alpha x} \; \ldots (1)$$

Note that $\frac{1}{f(D)}$ is the inverse operator to $f(D)$.

Operating both sides of (1) by $\frac{1}{f(D)}$ we have,

$$f(D) \cdot \frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(D)} f(\alpha) e^{\alpha x}$$

$$\Rightarrow e^{\alpha x} = \frac{1}{f(D)} f(\alpha) e^{\alpha x} \quad \left(\because f(D) \cdot \frac{1}{f(D)} = I\right)$$

$$\text{then } \frac{1}{f(\alpha)} e^{\alpha x} = \frac{1}{f(D)} e^{\alpha x}$$

Thus the *P.I.* is given by $\frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(\alpha)} e^{\alpha x}$ represented symbolically. …(2)

(2) holds when $f(\alpha) \neq 0$.

If $f(\alpha) = 0$ then $D = \alpha$ is a root of the characteristic equation for the differential equation $f(D) = 0 \Rightarrow D - \alpha$ is a factor of $f(D)$.

Let $f(D) = (D - \alpha)\, \theta(D)$, where $\theta(\alpha) \neq 0$ then

$$\frac{1}{f(D)} e^{\alpha x} = \frac{1}{(D-\alpha)\, \theta(D)} \cdot e^{\alpha x}$$

$$= \frac{1}{D-\alpha} \cdot \frac{1}{\theta(\alpha)} e^{\alpha x}$$

$$= \frac{1}{\theta(\alpha)} \frac{1}{D-\alpha} e^{\alpha x} \; \ldots (3)$$

$$\text{Put } \frac{1}{(D-\alpha)} e^{\alpha x} = y \Rightarrow (D-\alpha)y = e^{\alpha x}$$

$$\text{then } ye^{-\int \alpha\, dx} = \int e^{\alpha x} \cdot e^{-\int \alpha\, dx} \cdot dx$$

$$\text{i.e., } ye^{-\alpha x} = \int e^{\alpha x} e^{-\alpha x} dx \Rightarrow y = e^{\alpha x} x$$

Substituting in (3) we have

$$\frac{1}{f(D)} e^{\alpha x} = \frac{1}{\theta(\alpha)} x e^{\alpha x}$$

If further, $\theta(\alpha) = 0$, then $D = \alpha$ is a repeated root for $f(D) = 0$.

$$\text{Then } \frac{1}{f(D)} e^{\alpha x} = \frac{x^2}{2} e^{\alpha x}$$

Example 8.22 : Solve : $(D^2 + 5D + 6)y = 0$ or $y'' + 5y' + 6y = 0$

Solution : To find the *C.F.* solve the characteristic equation

$$p^2 + 5p + 6 = 0$$

$$\Rightarrow (p + 2)(p + 3) = 0 \Rightarrow p = -2 \text{ and } p = -3$$

The *C.F.* is $Ae^{-2x} + Be^{-3x}$.

Hence the general solution is $y = Ae^{-2x} + Be^{-3x}$ where A and B are arbitrary constants.

Example 8.23 : Solve : $(D^2 + 6D + 9)y = 0$

Solution : The characteristic equation is

$$p^2 + 6p + 9 = 0$$

$$\text{i.e., } (p + 3)^2 = 0 \Rightarrow p = -3, -3$$

The C.F. is $(Ax + B)e^{-3x}$

Hence the general solution is $y = (Ax + B)e^{-3x}$

where A and B are arbitrary constants.

Example 8.24 : Solve : $(D^2 + D + 1)y = 0$

Solution : The characteristic equation is $p^2 + p + 1 = 0$

$$\therefore p = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

Hence the general solution is $y = e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$

where A and B are arbitrary constant.

Example 8.25 : Solve : $(D^2 - 13D + 12)y = e^{-2x}$

Solution : The characteristic equation is $p^2 - 13p + 12 = 0$

$$\Rightarrow (p - 12)(p - 1) = 0 \quad \Rightarrow p = 12 \text{ and } 1$$

The C.F. is $Ae^{12x} + Be^{x}$

Particular integral $\quad P.I. = \dfrac{1}{D^2 - 13D + 12} e^{-2x}$

$$= \frac{1}{(-2)^2 - 13(-2) + 12} e^{-2x} = \frac{1}{4 + 26 + 12} e^{-2x}$$

$$= \frac{1}{42} e^{-2x}$$

Hence the general solution is $y = CF + PI \;\Rightarrow\; y = Ae^{12x} + Be^{x} + \frac{1}{42} e^{-2x}$

Example 8.26 : Solve : $(D^2 + 6D + 8)y = e^{-2x}$

Solution : The characteristic equation is $p^2 + 6p + 8 = 0$

$$\Rightarrow (p + 4)(p + 2) = 0 \;\Rightarrow\; p = -4 \text{ and } -2$$

The C.F. is $Ae^{-4x} + Be^{-2x}$

Particular integral $$P.I. = \frac{1}{D^2 + 6D + 8} e^{-2x} = \frac{1}{(D + 4)(D + 2)} e^{-2x}$$

Since $f(D) = (D + 2)\,\theta(D))$

$$= \frac{1}{\theta(-2)} xe^{-2x} = \frac{1}{2} xe^{-2x}$$

Hence the general solution is $y = Ae^{-4x} + Be^{-2x} + \frac{1}{2} xe^{-2x}$

Example 8.27 : Solve : $(D^2 - 6D + 9)y = e^{3x}$

Solution : The characteristic equation is $p^2 - 6p + 9 = 0$

i.e., $(p - 3)^2 = 0 \;\Rightarrow\; p = 3, 3$

The C.F. is $(Ax + B)e^{3x}$

Particular integral $$P.I. = \frac{1}{D^2 - 6D + 9} e^{3x}$$

$$= \frac{1}{(D - 3)^2} e^{3x} = \frac{x^2}{2} e^{3x}$$

Hence the general solution is $y = (Ax + B)e^{3x} + \frac{x^2}{2} e^{3x}$

Example 8.28 : Solve : $(2D^2 + 5D + 2)y = e^{-\frac{1}{2}x}$

Solution : The characteristic equation is $2p^2 + 5p + 2 = 0$

$$\therefore\; p = \frac{-5 \pm \sqrt{25 - 16}}{4} = \frac{-5 \pm 3}{4}$$

$$\Rightarrow \; p = -\frac{1}{2} \text{ and } -2$$

The C.F. is $Ae^{-\frac{1}{2}x} + Be^{-2x}$

Particular integral $\quad P.I. = \dfrac{1}{2D^2 + 5D + 2} e^{-\frac{1}{2}x} = \dfrac{1}{2\left(D + \frac{1}{2}\right)(D + 2)} e^{-\frac{1}{2}x}$

$$= \frac{1}{\theta\left(-\frac{1}{2}\right) . 2} xe^{-\frac{1}{2}x} = \frac{1}{3} x \, e^{-\frac{1}{2}x}$$

Hence the general solution is $y = Ae^{-\frac{1}{2}x} + Be^{-2x} + \frac{1}{3} x \, e^{-\frac{1}{2}x}$

Caution : In the above problem we see that while calculating the particular integral the coefficient of D expressed as factors is made unity.

(b) When X is of the form sin ax or cos ax.

Working rule :

Formula 1: Express $f(D)$ as function of D^2, say $\phi(D^2)$ and then replace D^2 by $-a^2$. If $\phi(-a^2) \neq 0$. Then we use the following result.

$$P.I. = \frac{1}{f(D)} \cos ax = \frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax$$

For example $\quad PI = \dfrac{1}{D^2 + 1} \cos 2x = \dfrac{1}{-2^2 + 1} \cos 2x = -\dfrac{1}{3} \cos 2x$

Formula 2 : Sometimes we cannot form $\phi(D^2)$. Then we shall try to get $\phi(D, D^2)$, that is, a function of D and D^2. In such cases we proceed as follows :

For example : $\quad P.I. = \dfrac{1}{D^2 - 2D + 1} \cos 3x$

$$= \frac{1}{-3^2 - 2D + 1} \cos 3x \qquad \text{Replace } D^2 \text{ by } -3^2$$

$$= \frac{-1}{2(D + 4)} \cos 3x$$

$$= \frac{-1}{2} \frac{D - 4}{D^2 - 4^2} \cos 3x \qquad \text{Multiply and divide by } D - 4$$

$$= \frac{-1}{2} \frac{1}{-3^2-4^2}(D-4)\cos 3x$$

$$= \frac{1}{50}(D-4)\cos 3x$$

$$= \frac{1}{50}[D\cos 3x - 4\cos 3x] = \frac{1}{50}[-3\sin 3x - 4\cos 3x]$$

Formula 3 : If $\phi(-a^2) = 0$ then we proceed as shown in the following example:

Example : $$P.I. = \frac{1}{\phi(D^2)}\cos ax = \frac{1}{D^2+a^2}\cos ax$$

$$= \frac{1}{(D+ia)(D-ia)}\cos ax$$

$$= \text{R.P.}\left[\frac{1}{(D+ia)(D-ia)}e^{iax}\right] = R.P.\left[\frac{1}{\theta(ia)}xe^{iax}\right]$$

$$= \text{Real part of}\left[\frac{xe^{iax}}{2ia}\right] \text{ as } \theta(ia) = 2ia$$

$$= \frac{-x}{2a}[\text{Real part of } i\,[\cos ax + i\sin ax]]$$

$$= \frac{-x}{2a}(-\sin ax) = \frac{x\sin ax}{2a}$$

Note : If $X = \sin ax$

Formula 1 : $\frac{1}{\phi(-a^2)}\sin ax$

Formula 2 : Same as $\cos ax$ method

Formula 3 : $\frac{1}{D^2+a^2}\sin ax = \text{I.P.}\left[\frac{1}{(D+ia)(D-ia)}e^{iax}\right] = \frac{-x}{2a}\cos ax$

Example 8.29 : Solve : $(D^2-4)y = \sin 2x$

Solution : The characteristic equation is $p^2 - 4 = 0 \Rightarrow p = \pm 2$

$$C.F. = Ae^{2x} + Be^{-2x} \;;$$

$$\text{P.I.} = \frac{1}{D^2-4}(\sin 2x) = \frac{1}{-4-4}(\sin 2x) = -\frac{1}{8}\sin 2x$$

Hence the general solution is $y = C.F. + P.I. \Rightarrow y = Ae^{2x} + Be^{-2x} - \frac{1}{8}\sin 2x$

Example 8.30 : Solve : $(D^2 + 4D + 13)y = \cos 3x$

Solution : The characteristic equation is $p^2 + 4p + 13 = 0$

$$p = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm i6}{2} = -2 \pm i3$$

$$C.F. = e^{-2x}(A \cos 3x + B \sin 3x)$$

$$P.I. = \frac{1}{D^2 + 4D + 13}(\cos 3x)$$

$$= \frac{1}{-3^2 + 4D + 13}(\cos 3x) = \frac{1}{4D+4}(\cos 3x)$$

$$= \frac{(4D-4)}{(4D+4)(4D-4)}(\cos 3x) = \frac{4D-4}{16D^2-16}(\cos 3x)$$

$$= \frac{4D-4}{-160}(\cos 3x) = \frac{1}{40}(3 \sin 3x + \cos 3x)$$

The general solution is y = C.$F.$ + $P.I.$

$$y = e^{-2x}(A \cos 3x + B \sin 3x) + \frac{1}{40}(3 \sin 3x + \cos 3x)$$

Example 8.31 : Solve $(D^2 + 9)y = \sin 3x$

Solution : The characteristic equation is $p^2 + 9 = 0 \;\Rightarrow\; p = \pm 3i$

$$C.F. = (A \cos 3x + B \sin 3x)$$

$$P.I. = \frac{1}{D^2+9} \sin 3x$$

$$= \frac{-x}{6} \cos 3x \quad \text{since } \frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax$$

Hence the solution is $y = C.F. + P.I.$

$$\text{i.e., } y = (A \cos 3x + B \sin 3x) - \frac{x \cos 3x}{6}$$

(c) When X is of the form x and x^2

Working rule : Take the P.I. as $c_0 + c_1x$ if $f(x) = x$ and $c_0 + c_1x + c_2x^2$ if $f(x) = x^2$. Since P.I. is also a solution of $(aD^2 + bD + c)_y = f(x)$, take $y = c_0 + c_1x$ or $y = c_0 + c_1x + c_2x^2$ according as $f(x) = x$ or x^2. By substituting y value and comparing the like terms, one can find c_0, c_1 and c_2.

Example 8.32 : Solve : $(D^2 - 3D + 2)y = x$

Solution : The characteristic equation is $p^2 - 3p + 2 = 0 \;\; \Rightarrow (p-1)(p-2) = 0$

$p = 1, 2$

The *C.F.* is $(Ae^x + Be^{2x})$

$$\text{Let } P.I. = c_0 + c_1 x$$

$\therefore\; c_0 + c_1x$ is also a solution.

$\therefore\; (D^2 - 3D + 2)(c_0 + c_1x) = x$

i.e., $(-3c_1 + 2c_0) + 2c_1x = x$

$$\Rightarrow 2c_1 = 1 \;\; \therefore\;\; c_1 = \frac{1}{2}$$

$$(-3c_1 + 2c_0) = 0 \;\Rightarrow\; c_0 = \frac{3}{4}$$

$$\therefore\;\; \text{P.I.} = \frac{x}{2} + \frac{3}{4}$$

Hence the general solution is $y = C.F. + P.I.$

$$y = Ae^x + Be^{2x} + \frac{x}{2} + \frac{3}{4}$$

Example 8.33 :

Solve : $(D^2 - 4D + 1)y = x^2$

Solution : The characteristic equation is $p^2 - 4p + 1 = 0$

$$\Rightarrow\; p = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$C.F. = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}$$

Let $P.I. = c_0 + c_1x + c_2x^2$

But P.I. is also a solution.

$\therefore\; (D^2 - 4D + 1)(c_0 + c_1x + c_2x^2) = x^2$

i.e., $(2c_2 - 4c_1 + c_0) + (-8c_2 + c_1)x + c_2x^2 = x^2$

$c_2 = 1$

$-8c_2 + c_1 = 0 \Rightarrow\; c_1 = 8$

$2c_2 - 4c_1 + c_0 = 0 \Rightarrow c_0 = 30$

P.I. $= x^2 + 8x + 30$

Hence the general solution is $y = C.F. + P.I.$

$$y = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x} + (x^2 + 8x + 30)$$

EXERCISE 8.5

Solve the following differential equations :

(1) $(D^2 + 7D + 12)y = e^{2x}$

(2) $(D^2 - 4D + 13)y = e^{-3x}$

(3) $(D^2 + 14D + 49)y = e^{-7x} + 4$

(4) $(D^2 - 13D + 12)y = e^{-2x} + 5e^x$

(5) $(D^2 + 1)\, y = 0$ when $x = 0$, $y = 2$ and when $x = \frac{\pi}{2}$, $y = -2$

(6) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$ when $x = \log 2$, $y = 0$ and when $x = 0$, $y = 0$

(7) $(D^2 + 3D - 4)\, y = x^2$

(8) $(D^2 - 2D - 3)y = \sin x\, \cos x$

(9) $D^2y = -9 \sin 3x$

(10) $(D^2 - 6D + 9)\, y = x + e^{2x}$

(11) $(D^2 - 1)y = \cos 2x - 2 \sin 2x$

(12) $(D^2 + 5)y = \cos^2 x$

(13) $(D^2 + 2D + 3)y = \sin 2x$

(14) $(3D^2 + 4D + 1)y = 3e^{-x/3}$

8.6 Applications :

In this section we solve problems on differential equations which have direct impact on real life situation. Solving of these types of problems involve

(i) Construction of the mathematical model describing the given situation

(ii) Seeking solution for the model formulated in (i) using the methods discussed earlier.

Illustration :

Let A be any population at time t. The rate of change of population is directly proportional to initial population i.e.,

$\frac{dA}{dt} \,\alpha\, A$ i.e., $\frac{dA}{dt} = kA$ where k is called the constant of proportionality

(1) If $k > 0$, we say that A grows exponentially with growth constant k (growth problem).

(2) If $k < 0$ we say that A decreases exponentially with decreasing constant k (decay problem).

In all the practical problems we apply the principle that the rate of change of population is directly proportional to the initial population i.e., $\frac{dA}{dt} \alpha A$ or $\frac{dA}{dt} = kA$

(Here k may be positive or negative depends on the problem). This linear equation can be solved in three ways i.e., (i) variable separable (ii) linear (using I.F.) (iii) by using characteristic equation with single root k. In all the ways we get the solution as $A = ce^{kt}$ where c is the arbitrary constant and k is the constant of proportionality. In general we have to find out c as well as k from the given data. Sometimes the value of k may be given directly as in 8.35. $\frac{dA}{dt}$ is directly given in 8.38.

Solution : $\frac{dA}{dt} = kA$

(i) $\frac{dA}{A} = kdt \quad \Rightarrow \quad \log A = kt + \log c$

$\Rightarrow \quad A = e^{kt + \log c} \Rightarrow A = ce^{kt}$

(ii) $\frac{dA}{dt} - kA = 0$ is linear in A

$$I.F. = e^{-kt}$$

$$Ae^{-kt} = \int e^{-kt} O\, dt + c \Rightarrow Ae^{-kt} = c$$

$$A = ce^{kt}$$

(iii) $(D - k)A = 0$

Chr. equation is $p - k = 0 \Rightarrow p = k$

The C.F. is ce^{kt}

But there is no $P.I.$

$\therefore A = ce^{kt}$

(iv) In the case of Newton's law of cooling (i.e., the rate of change of temperature is proportional to the difference in temperatures) we get the equation as

$$\frac{dT}{dt} = k(T - S)$$

[T– cooling object temperature, S – surrounding temperature]

$$\frac{dT}{T - S} = kdt \Rightarrow \log(T - S) = kt + \log c \Rightarrow T - S = ce^{kt}$$

$$\Rightarrow \quad T = S + ce^{kt}$$

Example 8.34 : In a certain chemical reaction the rate of conversion of a substance at time t is proportional to the quantity of the substance still untransformed at that instant. At the end of one hour, 60 grams remain and at the end of 4 hours 21 grams. How many grams of the substance was there initially?

Solution :

Let A be the substance at time t

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \;\Rightarrow\; A = ce^{kt}$$

$$\text{When } t = 1, \quad A = 60 \Rightarrow ce^{k} = 60 \qquad \ldots (1)$$

$$\text{When } t = 4, A = 21 \;\Rightarrow ce^{4k} = 21 \qquad \ldots (2)$$

$$(1) \Rightarrow c^4 e^{4k} = 60^4 \qquad \ldots (3)$$

$$\frac{(3)}{(2)} \;\Rightarrow\; c^3 = \frac{60^4}{21} \;\Rightarrow\; c = 85.15 \text{ (by using log)}$$

Initially i.e., when $t = 0$, $A = c = 85.15$ gms (app.)

Hence initially there was 85.15 gms (approximately) of the substance.

Example 8.35 : A bank pays interest by continuous compounding, that is by treating the interest rate as the instantaneous rate of change of principal. Suppose in an account interest accrues at 8% per year compounded continuously. Calculate the percentage increase in such an account over one year. [Take $e^{.08} \approx 1.0833$]

Solution : Let A be the principal at time t

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \;\Rightarrow \frac{dA}{dt} = 0.08\,A, \text{ since } k = 0.08$$

$$\Rightarrow\; A(t) = ce^{0.08t}$$

$$\text{Percentage increase in 1 year} = \frac{A(1) - A(0)}{A(0)} \times 100$$

$$= \left(\frac{A(1)}{A(0)} - 1\right) \times 100 = \left(\frac{c.\,e^{0.08}}{c} - 1\right) \times 100 = 8.33\%$$

Hence percentage increase is 8.33%

Example 8.36 :

The temperature T of a cooling object drops at a rate proportional to the difference $T - S$, where S is constant temperature of surrounding medium. If initially $T = 150°C$, find the temperature of the cooling object at any time t.

Solution :

Let T be the temperature of the cooling object at any time t

$\frac{dT}{dt} \propto (T - S) \Rightarrow \frac{dT}{dt} = k(T - S) \Rightarrow T - S = ce^{kt}$, where k is negative

$\Rightarrow \; T = S + ce^{kt}$

When $t = 0$, $T = 150 \Rightarrow 150 = S + c \Rightarrow c = 150 - S$

$\therefore$ The temperature of the cooling object at any time is

$T = S + (150 - S)e^{kt}$

Note : Since k is negative, as t increases T decreases.

It is a decay problem. Instead of k one may take $-k$ where $k > 0$. Then the answer is $T = S + (150 - S)e^{-kt}$. Again, as t increases T decreases.

Example 8.37 : For a postmortem report, a doctor requires to know approximately the time of death of the deceased. He records the first temperature at 10.00 a.m. to be 93.4°F. After 2 hours he finds the temperature to be 91.4°F. If the room temperature (which is constant) is 72°F, estimate the time of death. (Assume normal temperature of a human body to be 98.6°F).

$\left[\log_e \frac{19.4}{21.4} = -0.0426 \times 2.303 \text{ and } \log_e \frac{26.6}{21.4} = 0.0945 \times 2.303\right]$

Solution :

Let T be the temperature of the body at any time t

By Newton's law of cooling $\frac{dT}{dt} \propto (T - 72)$ since $S = 72°F$

$$\frac{dT}{dt} = k(T - 72) \Rightarrow T - 72 = ce^{kt}$$

$$\text{or} \quad T = 72 + ce^{kt}$$

At $t = 0$, $T = 93.4 \Rightarrow c = 21.4$ [First recorded time 10 a.m. is $t = 0$]

$$\therefore T = 72 + 21.4e^{kt}$$

When $t = 120$, $T = 91.4 \Rightarrow e^{120k} = \frac{19.4}{21.4} \Rightarrow k = \frac{1}{120}\log_e\left(\frac{19.4}{21.4}\right)$

$= \frac{1}{120}(-0.0426 \times 2.303)$

Let t_1 be the elapsed time after the death.

When $t = t_1$; $T = 98.6 \Rightarrow 98.6 = 72 + 21.4\, e^{kt_1}$

$$\Rightarrow t_1 = \frac{1}{k}\ \log_e\left(\frac{26.6}{21.4}\right) = \frac{-120 \times 0.0945 \times 2.303}{0.0426 \times 2.303} = -266 \text{ min}$$

[For better approximation the hours converted into minutes]

i.e., 4 hours 26 minutes before the first recorded temperature.

The approximate time of death is 10.00 hrs – 4 hours 26 minutes.

$\therefore$ Approximate time of death is 5.34 A.M.

Note : Since it is a decay problem, we can even take $\frac{dT}{dt} = -k(T - 72)$ where $k > 0$. The final answer will be the same.

Example 8.38 : A drug is excreted in a patients urine. The urine is monitored continuously using a catheter. A patient is administered 10 mg of drug at time $t = 0$, which is excreted at a Rate of $-3t^{1/2}$ mg/h.

(i) What is the general equation for the amount of drug in the patient at time $t > 0$?

(ii) When will the patient be drug free?

Solution :

(i) Let A be the quantum of drug at any time t

The drug is excreted at a rate of $-3t^{\frac{1}{2}}$

$$\text{i.e., } \frac{dA}{dt} = -3t^{\frac{1}{2}} \Rightarrow A = -2t^{\frac{3}{2}} + c$$

$$\text{When } t = 0,\ A = 10 \Rightarrow c = 10$$

At any time t $\qquad A = 10 - 2t^{\frac{3}{2}}$

(ii) For drug free, $A = 0 \Rightarrow 5 = t^{\frac{3}{2}} \Rightarrow t^3 = 25 \Rightarrow t = 2.9$ hours.

Hence the patient will be drug free in 2.9 hours or 2 hours 54 min.

Example 8.39 :

The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour. Show that the number of bacteria at the end of five hours will be 3^5 times of the population at initial time.

Solution : Let A be the number of bacteria at any time t

$$\frac{dA}{dt}\ \alpha\ A \Rightarrow \frac{dA}{dt} = kA \Rightarrow A = ce^{kt}$$

Initially, i.e., when $t = 0$, assume that $A = A_0$

$$\therefore\ A_0 = ce^{\circ} = c$$

$$\therefore\ A = A_0 e^{kt}$$

$$\text{when } t = 1,\ A = 3A_0 \Rightarrow 3A_0 = A_0 e^k \Rightarrow e^k = 3$$

$$\text{When } t = 5,\ A = A_0 e^{5k} = A_0 (e^k)^5 = 3^5 .\ A_0$$

$\therefore$ The number of bacteria at the end of 5 hours will be 3^5 times of the number of bacteria at initial time

EXERCISE 8.6

(1) Radium disappears at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years, how much will remain at the end of 100 years. [Take A_0 as the initial amount].

(2) The sum of Rs. 1000 is compounded continuously, the nominal rate of interest being four percent per annum. In how many years will the amount be twice the original principal? ($\log_e 2 = 0.6931$).

(3) A cup of coffee at temperature 100°C is placed in a room whose temperature is 15°C and it cools to 60°C in 5 minutes. Find its temperature after a further interval of 5 minutes.

(4) The rate at which the population of a city increases at any time is proportional to the population at that time. If there were 1,30,000 people in the city in 1960 and 1,60,000 in 1990 what population may be anticipated in 2020. $\left[\log_e \left(\frac{16}{13}\right) = .2070 \ ; \ e^{.42} = 1.52\right]$

(5) A radioactive substance disintegrates at a rate proportional to its mass. When its mass is 10 mgm, the rate of disintegration is 0.051 mgm per day. How long will it take for the mass to be reduced from 10 mgm to 5 mgm. [$\log_e 2 = 0.6931$]

9. DISCRETE MATHEMATICS

Discrete Mathematics deals with several selected topics in Mathematics that are essential to the study of many Computer Science areas. Since it is very difficult to cover all the topics, only two topics, namely "Mathematical Logic", and "Groups" have been introduced. These topics will be very much helpful to the students in certain practical applications related to Computer Science.

9.1. Mathematical Logic : Introduction :

Logic deals with all types of reasonings. These reasonings may be legal arguments or mathematical proofs or conclusions in a scientific theory. Aristotle (384 – 322 BC) wrote the first treatise on logic. Gottfried Leibnitz framed the idea of using symbols in logic and this idea was realised in the nineteenth century by George Boole and Augustus De'Morgan.

Logic is widely used in many branches of sciences and social sciences. It is the theoretical basis for many areas of Computer Science such as digital logic circuit design, automata theory and artificial intelligence.

We express our thoughts through words. Since words have many associations in every day life, there are chances of ambiguities to appear. In order to avoid this, we use symbols which have been clearly defined. Symbols are abstract and neutral. Also they are easy to write and manipulate. This is because the mathematical logic which we shall study is also called symbolic logic.

9.1.1 Logical statement or Proposition :

A statement or a proposition is a sentence which is either true or false but not both.

A sentence which is both true and false simultaneously is not a statement, rather it is a paradox.

Example 1 :

(a) Consider the following sentences :

(i) Chennai is the capital of Tamilnadu.

(ii) The earth is a planet.

(iii) Rose is a flower.

Each of these sentences is true and so each of them is a statement.

(b) Consider the following sentences :

(iv) Every triangle is an isosceles triangle.

(v) Three plus four is eight

(vi) The sun is a planet.

Each of these sentences is false and so each of them is a statement.

Example 2 : Each of the sentences

(vii) Switch on the light.

(viii) Where are you going?

(ix) May God bless you with success.

(x) How beautiful Taj Mahal is!

cannot be assigned true or false and so none of them is a statement. In fact, (vii) is a command, (viii) is a question (ix) is an optative and (x) is exclamatory.

Truth value of a statement :

The truth or falsity of a statement is called its truth value. If a statement is true, we say that its truth value is TRUE or T and if it is false, we say that its truth value is FALSE or F.

All the statements in Example 1(a) have the truth value T, while all the statements in Example 1 (b) have the truth value F.

Simple statements :

A statement is said to be simple if it cannot be broken into two or more statements. All the statements in (a) and (b) of Example 1 are simple statements.

Compound statements :

If a statement is the combination of two or more simple statements, then it is said to be a compound statement.

Example : It is raining and it is cold.

This is a compound statement and it is a combination of two simple statements "It is raining", "It is cold".

Simple statements which on combining form compound statements are called sub-statements or component statements of the compound statement.

The fundamental property of a compound statement is that its truth value is completely determined by the truth values of its sub-statements together with the way in which they are combined to form the compound statement.

Basic logical connectives

The words which combine simple statements to form compound statements are called connectives. We use the connectives 'and', 'or', etc., to form new statements by combining two or more statements. But the use of these connectives in English language is not always precise and unambiguous. Hence it is necessary to define a set of connectives with definite meanings in the language of logic, called object language. Three basic connectives are conjunction which corresponds to the English word ‘and’, ‘disjunction’ which corresponds to the word ‘or’ ‘negation’ which corresponds to the word ‘not’.

We use the symbol “$\wedge$” to denote conjunction, “$\vee$” to denote disjunction and “ ~ ” to denote negation.

Conjunction :

If two simple statements p and q are connected by the word ‘and’, then the resulting compound statement ‘p and q’ is called the conjunction of p and q and is written in the symbolic form as ‘$p \wedge q$’.

Example 1 : Form the conjunction of the following simple statements

p : Ram is intelligent.

q : Ravi is handsome.

$p \wedge q$: Ram intelligent and Ravi is handsome.

Example 2 : Convert the following statement into symbolic form :

‘Usha and Mala are going to school’.

the given statement can be rewritten as :

‘Usha is going to school’, and

‘Mala is going to school’.

Let p : Usha is going to school.

q : Mala is going to school.

The given statement in symbolic form is $p \wedge q$.

Rule : (A_1) The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T.

(A_2) The statement $p \wedge q$ has the truth value F whenever either p or q or both have the truth value F.

Example : Write the truth value of each of the following statements :

(i) Ooty is in Tamilnadu and 3 + 4 = 8

(ii) Ooty is in Tamilnadu and 3 + 4 = 7

(iii) Ooty is in Kerala and 3 + 4 = 7

(iv) Ooty is in Kerala and 3 + 4 = 8

In (i) the truth value of the statement 3 + 4 = 8 is F. $\therefore$ By (A_2)

(i) has the truth value F.

In (ii) both the sub-statements have truth value T and hence by (A_1). (ii) has truth value T.

The truth values of (iii) and (iv) are F.

Disjunction :

If two simple statements p and q are connected by the word 'or', then the resulting compound statement 'p or q' is called the disjunction of p and q and is written in symbolic form as $p \vee q$.

Example : Form the disjunction of the following simple statements :

p : John is playing cricket.

q : There are thirty students in the class room.

$p \vee q$: John is playing cricket or there are thirty students in the class room.

Example : Convert the following statement into symbolic form.

"5 is a positive integer or a square is a rectangle".

Let p : 5 is a positive integer.

q : A square is a rectangle.

The given statement in symbolic form is $p \vee q$.

Rule : (A_3) The statement $p \vee q$ has the truth value F whenever both p and q have the truth value F.

(A_4) The statement $p \vee q$ has the truth value T whenever either p or q or both have the truth value T.

Example :

(i) Chennai is in India or $\sqrt{2}$ is an integer.

(ii) Chennai is in India or $\sqrt{2}$ is an irrational number.

(iii) Chennai is in China or $\sqrt{2}$ is an integer.

(iv) Chennai is in China or $\sqrt{2}$ is an irrational number.

By (A_4), we see that the truth values of (i), (ii) and (iv) are T and by (A_3), the truth value of (iii) is F.

Negation :

The negation of a statement is generally formed by introducing the word 'not' at some proper place in the statement or by prefixing the statement with 'It is not the case that' or 'It is false that'.

If p denotes a statement, then the negation of p is written as $\sim p$ or $\rceil p$. We use the symbol $\sim p$ to denote the negation of p.

Rule : (A_5) If the truth value of p is T then the truth value of $\sim p$ is F. Also, if the truth value of p is F, then the truth value of $\sim p$ is T.

Example :

p : All men are wise.

$\sim p$: Not all men are wise. (or)

$\sim p$: It is not the case that all men are wise (or)

$\sim p$: It is false that all men are wise.

Note : Negation is called a connective although it does not combine two or more statements. It only modifies a statement.

EXERCISE 9.1

Find out which of the following sentences are statements and which are not? Justify your answer.

(1) All natural numbers are integers.

(2) A square has five sides.

(3) The sky is blue.

(4) How are you?

(5) $7 + 2 < 10$.

(6) The set of rational numbers is finite.

(7) How beautiful you are!

(8) Wish you all success.

(9) Give me a cup of tea.

(10) 2 is the only even prime.

Write down the truth value (T or F) of the following statements :

(11) All the sides of a rhombus are equal in length.

(12) $1 + \sqrt{8}$ is an irrational number.

(13) Milk is white.

(14) The number 30 has four prime factors.

(15) Paris is in France.

(16) Sin x is an even function.

(17) Every square matrix is non-singular.

(18) Jupiter is a planet.

(19) The product of a complex number and its conjugate is purely imaginary.

(20) Isosceles triangles are equilateral.

(21) Form the conjunction and the disjunction of

(i) p : Anand reads newspaper, q : Anand plays cricket.

(ii) p : I like tea. q : I like ice-cream.

(22) Let p be "Kamala is going to school" and q be "There are twenty students in the class ". Give a simple verbal sentence which describes each of the following statements :

(i) $p \vee q$ (ii) $p \wedge q$ (iii) $\sim p$ (iv) $\sim q$ (v) $\sim p \vee q$

(23) Translate each of the following compound statements into symbolic form :

(i) Rose is red and parrot is a bird.

(ii) Suresh reads 'Indian Express' or 'The Hindu'.

(iii) It is false that the mangoes are sweet.

(iv) $3 + 2 = 5$ and Ganges is a river.

(v) It is false that sky is not blue.

(24) If p stands for the statement "Sita likes reading" and q for the statement "Sita likes playing' what does $\sim p \wedge \sim q$ stand for?

(25) Write negation of the each of the following statements :

(i) $\sqrt{5}$ is an irrational number.

(ii) Mani is sincere and hardworking.

(iii) This picture is good or beautiful.

9.1.2 Truth tables :

A table that shows the relation between the truth values of a compound statement and the truth values of its sub-statements is called the truth table. A truth table consists of rows and columns. The initial columns are filled with the possible truth values of the sub-statements and the last column is filled with the truth values of the compound statement on the basis of the truth values of the sub-statements written in the initial columns. If the compound statement is made up of n sub-statements, then its truth table will contain 2^n rows.

Example 9.1 : Construct the truth table for $\sim p$

Solution: The statement $\sim p$ consists of only one simple statement p. Therefore, its truth table will contain $2^1 (= 2)$ rows.

Also we know that if p has the truth value T then $\sim p$ has the truth value F and if p has the truth value F, then $\sim p$ has the truth value T. Thus the truth table for $\sim p$ is as given below :

Truth table for $\sim p$

p	$\sim p$
T	F
F	T

Example 9.2 : Construct the truth table for $p \vee (\sim p)$.

Solution: The compound statement $p\vee$ $(\sim p)$ consists of only one single statement p. Therefore its truth table will contain 2^1(= 2) rows.

In the first column, enter all possible truth values of p.

In the second column, enter the truth values of $\sim p$ based on the corresponding truth values of p. Finally, in the last column, enter the truth values of $p \vee (\sim p)$, using (A_4).

Truth table for $p\vee(\sim p)$

p	$\sim p$	$p\vee(\sim p)$
T	F	T
F	T	T

Example 9.3 : Construct the truth table for $p \wedge q$.

Solution: The compound statement $p \wedge q$ consists of two simple statements p and q. Therefore, there must be 2^2(= 4) rows in the truth table of $p \wedge q$. Now enter all possible truth values of statements p and q namely *TT*, *TF*, *FT* and *FF* in the first two columns of the truth table.

Using (A_1) and (A_2), enter the truth values of $p \wedge q$ in the final column based on the corresponding truth values of p and q in the first two columns.

Truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note : Similarly, by using (A_3) and (A_4) we can construct the truth table for $p \vee q$, as given below :

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 9.4 : Construct the truth table for the following statements :

(i) $((\sim p) \vee (\sim q))$ (ii) $\sim ((\sim p) \wedge q)$

(iii) $(p \vee q) \wedge (\sim q)$ (iv) $\sim ((\sim p) \wedge (\sim q))$

Solution:

(i) **Truth table for $((\sim p) \vee (\sim q))$**

p	q	$\sim p$	$\sim q$	$((\sim p) \vee (\sim q))$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(ii) **Truth table for $\sim ((\sim p) \wedge q)$**

p	q	$\sim p$	$(\sim p) \wedge q$	$\sim ((\sim p) \wedge q)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

(iii) **Truth table for $(p \vee q) \wedge (\sim q)$**

p	q	$p \vee q$	$\sim q$	$(p \vee q) \wedge (- q)$
T	T	T	F	F
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

(iv) **Truth table for $\sim ((\sim p) \wedge (\sim q))$**

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$	$\sim ((\sim p) \wedge (\sim q))$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

Example 9.5 : Construct the truth table for $(p \wedge q) \vee (\sim r)$

Solution: The compound statement $(p \wedge q) \vee (\sim r)$ consists of three simple statements p, q and r. Therefore, there must be $2^3 (= 8)$ rows in the truth table of $(p \wedge q) \vee (\sim r)$. The truth value of p remains at the same value of T or F for each of four consecutive assignments of logical values. The truth value of q remains at T or F for two assignments and that of r remains at T or F for one assignment.

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee (\sim r)$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Example 9.6 : Construct the truth table for $(p \vee q) \wedge r$

Solution:

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

EXERCISE 9.2

Construct the truth tables for the following statements :

(1) $p \vee (\sim q)$

(2) $(\sim p) \wedge (\sim q)$

(3) $\sim (p \vee q)$

(4) $(p \vee q) \vee (\sim p)$

(5) $(p \wedge q) \vee (\sim q)$

(6) $\sim (p \vee (\sim q))$

(7) $(p \wedge q) \vee [\sim (p \wedge q)]$

(8) $(p \wedge q) \vee (\sim q)$

(9) $(p \vee q) \vee r$

(10) $(p \wedge q) \vee r$

Logical Equivalence :

Two compound statements A and B are said to be logically equivalent or simply equivalent, if they have identical last columns in their truth tables.

In this case we write $A \equiv B$.

Example 9.7 : Show that $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

Solution:

Truth table for ~ ($p \vee q$)

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Truth table for (($\sim p$) $\wedge$ ($\sim q$))

p	q	$\sim p$	$\sim q$	$((\sim p) \wedge (\sim q))$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The last columns are identical. $\therefore \sim (p \vee q) \equiv ((\sim p) \wedge (\sim q))$

Negation of a negation :

Negation of a negation of a statement is the statement itself. Equivalently we write $\sim (\sim p) \equiv p$

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

In the truth table, the columns corresponding to p and $\sim (\sim p)$ are identical. Hence p and $\sim (\sim p)$ are logically equivalent.

Example 9.8 : Verify $\sim (\sim p) \equiv p$ for the statement p : the sky is blue.

Solution:

p : The sky is blue

$\sim p$: The sky is not blue

$\sim (\sim p)$: It is not the case that the sky is not blue or
It is false that the sky is not blue or
The sky is blue

Conditional and bi-conditional statements :

In Mathematics, we frequently come across statements of the form "If p then q". Such statements are called conditional statements or implications. They are denoted by $p \rightarrow q$, read as 'p implies q'. The conditional $p \rightarrow q$ is false only if p is true and q is false. Accordingly, if p is false then $p \rightarrow q$ is true regardless of the truth value of q.

Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If p and q are two statements, then the compound statement $p \to q$ and $q \to p$ is called a bi-conditional statement and is denoted by $p \leftrightarrow q$, read as p if and only if q. $p \leftrightarrow q$ has the truth value T whenever p and q have the same truth values; otherwise it is F.

Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

9.1.3 Tautologies :

A statement is said to be a tautology if the last column of its truth table contains only T, i.e., it is true for all logical possibilities.

A statement is said to be a contradiction if the last column of its truth table contains only F, i.e., it is false for all logical possibilities.

Example 9.9 : (i) $p \vee (\sim p)$ is a tautology.   (ii) $p \wedge (\sim p)$ is a contradiction

Solution:

(i) **Truth table for $p \vee (\sim p)$**

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

The last column contains only T. $\therefore$ $p \vee (\sim p)$ is a tautology.

(ii) **Truth table for $p \wedge (\sim p)$**

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

The last column contains only F. $\therefore$ $p \wedge (\sim p)$ is a contradiction.

Example 9.10 : (i) Show that $((\sim p) \vee (\sim q)) \vee p$ is a tautology.

(ii) Show that $((\sim q) \wedge p) \wedge q$ is a contradiction.

Solution:

(i) **Truth table for $((\sim p) \vee (\sim q)) \vee p$**

p	q	$\sim p$	$\sim q$	$(\sim p) \vee (\sim q)$	$((\sim p) \vee (\sim q)) \vee p$
T	T	F	F	F	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

The last column contains only T. $\therefore$ $((\sim p) \vee (\sim q)) \vee p$ is a tautology.

(ii) **Truth table for $((\sim q) \wedge p) \wedge q$**

p	q	$\sim q$	$(\sim q) \wedge p$	$((\sim q) \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

The last column contains only F. $\therefore$ $((\sim q) \wedge p) \wedge q$ is a contradiction.

Example 9.11 : Use the truth table to determine whether the statement $((\sim p) \vee q) \vee (p \wedge (\sim q))$ is a tautology.

Solution:

Truth table for $((\sim p) \vee q) \vee (p \wedge (\sim q))$

p	q	$\sim p$	$\sim q$	$(\sim p) \vee q$	$p \wedge (\sim q)$	$((\sim p) \vee q) \vee (p \wedge (\sim q)$
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

The last column contains only T. $\therefore$ The given statement is a tautology.

EXERCISE 9.3

(1) Use the truth table to establish which of the following statements are tautologies and which are contradictions.

(i) $((\sim p) \wedge q) \wedge p$

(ii) $(p \vee q) \vee (\sim (p \vee q))$

(iii) $(p \wedge (\sim q)) \vee ((\sim p) \vee q)$

(iv) $q \vee (p \vee (\sim q))$

(v) $(p \wedge (\sim p)) \wedge ((\sim q) \wedge p)$

(2) Show that $p \to q \equiv (\sim p) \vee q$

(3) Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

(4) Show that $p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)$

(5) Show that $\sim(p \wedge q) \equiv ((\sim p) \vee (\sim q))$

(6) Show that $p \to q$ and $q \to p$ are not equivalent.

(7) Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

9.2 Groups :

9.2.1 Binary Operation :

We know that the addition of any two natural numbers is a natural number, the product of any two natural numbers is also a natural number. Each of these operations associates with the two given numbers, a third number, their sum in the case of addition, and their product in the case of multiplication. In this section we are going to deal with the notion of a binary operation or a binary composition on a set which is nothing but a generalisation of the usual addition and usual multiplication on the number systems.

Definition :

A binary operation * on a non-empty set S is a rule, which associates to each ordered pair (a, b) of elements a, b in S an element $a * b$ in S. Thus a binary operation * on S is just a map, $* : S \times S \to S$ by $(a, b) \to a * b$.

Where we denote by $a * b$, the image of (a, b) in S under *.

From the definition we see that, if * is a binary operation on S then $a, b \in S \Rightarrow a * b \in S$.

In this case, we also say that S is closed under *. This property is known as the "closure axiom" or "closure property".

List of symbols used in this chapter :

N	-	The set of all natural numbers.
Z	-	The set of all integers.
W	-	The set of all non-negative integers (whole numbers).
E	-	The set of all even integers.
O	-	The set of all odd integers.
Q	-	The set of all rational numbers.
R	-	The set of all real numbers.
C	-	The set of all complex numbers.
$Q - \{0\}$	-	The set of all non-zero rational numbers.
$R - \{0\}$	-	The set of all non-zero real numbers.
C - }0}	-	The set of all non-zero complex numbers.
$\forall$	-	for every
$\exists$	-	there exists
$\ni$	-	such that

Illustrative examples :

The usual addition + is a binary operation on N.

Since $a, b \in N \Rightarrow a + b \in N$. i.e., N is closed under +.

But the usual subtraction is not binary on N. Since $2, 5 \in N$,

but $2 - 5 = -3 \notin N$.

$\therefore$ N is not closed under subtraction.

At the same time, we see that $-$ is a binary operation on Z. From this we see that, an operation becoming binary or not binary depends on the set. The following table gives which number systems are closed under the usual algebraic operations, namely addition, subtraction, multiplication and division denoted by +, –, . , ÷ respectively.

Operations \ Number Systems	N	Z	Q	R	C	$Q - \{0\}$	$R - \{0\}$	$C - \{0\}$
+	binary	binary	binary	binary	binary	not binary	not binary	not binary
–	not binary	binary	binary	binary	binary	not binary	not binary	not binary
.	binary	binary	binary	binary	binary	binary	binary	binary
÷	not binary	not binary	not binary	not binary	not binary	binary	binary	binary

Apart from the usual algebraic operations, some new operations on the number systems can also be defined. For example, consider the operation * on N defined by $a * b = a^b$.

It is clear that * is binary on N, $\because a, b \in N \Rightarrow a * b = a^b \in N$.

Some more facts about binary operations :

(1) Let the set S be R or any subset of real number system.

Define * as (i) $a * b$ = minimum of $\{a, b\}$

(ii) $a * b$ = maximum of $\{a, b\}$

(iii) $a * b = a$

(iv) $a * b = b$

All the above operations (*) are binary operations on the corresponding sets.

(2) $(N, *)$

$*$ is defined as $a * b = ab + 5$. Since ab and 5 are natural numbers, $ab + 5$ is also a natural number. $\therefore$ $*$ is a binary operation on N.

On the other hand, the operation $*$ defined by $a * b = ab - 5$ is not binary on N because $2 * 1 = (2)(1) - 5 = -3 \notin N$.

(3) $(Z, *)$, where $*$ is defined by, $a * b = a^b$, is not a binary operation on z.

Since take $a = 2$, $b = -1$

$a^b = 2^{-1} = \frac{1}{2} \notin Z$

Note that $*$ is also not a binary operaton on $R - \{0\}$

because take $a = -1$, $b = \frac{1}{2}$ $\quad ab = (-1)^{1/2} \notin R - \{0\}$

(4) $(R, *)$

Define $a * b = a + b + ab$

Clearly $*$ is a binary operation on R since $a + b$ and ab are real numbers and their sum is also a real number.

(5) $(O, +)$

Addition is not a binary operation on the set of odd integers, since addition of two odd integers is not odd.

(6) $(O, .)$

Multiplication is a binary operation on the set of odd integers. Since product of two odd integers is an odd integer.

(7) Matrix addition is a binary operation on the set of $m \times n$ matrices. Since sum of two $m \times n$ matrices is again an $m \times n$ matrix.

(8) Matrix addition is not a binary operation on the set of $n \times n$ singular matrices as well as on the set of $n \times n$ non-singular matrices. Because, sum of two non-singular matrices need not be non-singular and sum of two singular matrices need not be singular.

(9) Matrix multiplication is a binary operation on the set of singular matrices as well as on the set of non-singular matrices.

(10) Cross product is a binary operation on the set of vectors, but dot product is not a binary operation on the set of vectors.

Multiplication table for a binary operation

Any binary operation $*$ on a finite set $S = \{a_1, a_2 \ldots a_n\}$ can be described by means of multiplication table. This table consists of 'n' rows and 'n' columns. Place each element of S at the head of one row and one column, usually taking them in the same order for columns as for rows. The operator $*$ is placed at the left hand top corner. The $n \times n = n^2$ spaces can be filled by writing $a_i * a_j$ in the space common to the ith row and the jth column of the table.

*	a_1	a_2	....	a_j	...
a_1					
. . . .					
a_i				$a_i * a_j$	
. . .					

This table is also known as **Cayley's table** or composition table. In the next section we will see that these composition tables are very much helpful in exhibiting finite groups.

9.2.2 Groups :

Given any non-empty set S, the possibility of combining two of its elements to get yet another element of S endows S with an algebraic structure. A non-empty set S together with a binary operation * is called an algebraic structure. Group is the simplest of all algebraic structures. It is the one operational algebraic system. The study of groups was started in the nineteenth century in connection with the solution of equations. The concept of group arises not only in Mathematics but also in other fields like Physics, Chemistry and Biology.

Definition :

A non-empty set G, together with an operation * i.e., $(G, *)$ is said to be a group if it satisfies the following axioms

(1) **Closure axiom** : $a, b \in G \Rightarrow a * b \in G$

(2) **Associative axiom** : $\forall a, b, c \in G, (a * b) * c = a * (b * c)$

(3) **Identity axiom** : There exists an element $e \in G$ such that $a * e = e * a = a, \forall a \in G$.

(4) **Inverse axiom** : $\forall a \in G$ there exists an element $a^{-1} \in G$ such that $a^{-1} * a = a * a^{-1} = e$.

e is called the identity element of G and a^{-1} is called the inverse of a in G.

Definition (Commutative property) :

A binary operation * on a set S is said to be commutative if $a * b = b * a \ \forall \ a, b \in S$

Definition :

If a group satisfies the commutative property then it is called an abelian group or a commutative group, otherwise it is called a non-abelian group.

Note (1) :

If the operation * is a binary operation, the closure axiom will be satisfied automatically.

Note (2) :

We shall often use the same symbol G to denote the group and the underlying set.

Order of a group :

The order of a group is defined as the number of distinct elements in the underlying set.

If the number of elements is finite then the group is called a finite group and if the number of elements is infinite then the group is called an infinite group. The order of a group G is denoted by $o(G)$.

Definition :

A non-empty set S with an operation * i.e., $(S, *)$ is said to be a semi-group if it satisfies the following axioms.

(1) **Closure axiom** : $a, b \in S \Rightarrow a * b \in S$

(2) **Associative axiom** : $(a * b) * c = a * (b * c), \forall\, a, b, c \in S.$

Definition :

A non-empty set M with an operation * i.e., $(M, *)$ is said to be a monoid if it satisfies the following axioms :

(1) **Closure axiom** : $a, b \in M \Rightarrow a * b \in M$

(2) **Associative axiom** : $(a * b) * c = a * (b * c)\ \forall a, b, c \in M$

(3) **Identity axiom** : There exists an element $e \in M$ such that $a * e = e * a = a, \forall a \in M.$

$(N, +)$ is a semi-group but it is not a monoid, because the identity element $O \notin N$.

$(N, *)$ where * is defined by $a * b = a^b$ is not a semi-group, because, consider

$$(2 * 3) * 4 = 2^3 * 4 = 8^4 = 2^{12} \text{ and}$$
$$2 * (3 * 4) = 2 * 3^4 = 2 * 81 = 2^{81}$$

$\therefore\ (2 * 3) * 4 \neq 2 * (3 * 4)$ i.e., associative axiom is not satisfied.

$(Z, .)$ is a monoid. But it is not a group, because, the inverse axiom is not satisfied. ($5 \in Z$, but $\frac{1}{5} \notin Z$). $(Z, +)$ and $(Z, .)$ are semi-groups as well as monoids. From the definitions, it is clear that every group is a monoid.

Example 9.12 : Prove that $(Z, +)$ is an infinite abelian group.

Solution:

(i) **Closure axiom** **:** We know that sum of two integers is again an integer.

(ii) **Associative axiom** **:** Addition is always associative in Z

i.e., $\forall a, b, c \in Z,\ (a + b) + c = a + (b + c)$

(iii) **Identity axiom** **:** The identity element $O \in Z$ and it satisfies

$O + a = a + O = a, \forall\ a \in Z$

Identity axiom is true.

(iv) **Inverse axiom** **:** For every $a \in Z$, $\exists$ an element $-a \in Z$ such that $-a + a = a + (-a) = 0$

$\therefore$ Inverse axiom is true. $\therefore$ $(Z, +)$ is a group.

(v) $\forall\ a, b \in Z,\ a + b = b + a$

$\therefore$ addition is commutative. $\therefore$ $(Z, +)$ is an abelian group.

(vi) Since Z is an infinite set $(Z, +)$ is infinite abelian group.

Example 9.13 : Show that $(R - \{0\}, \textbf{.})$ is an infinite abelian group. Here '.' denotes usual multiplication.

Solution:

(i) **Closure axiom** **:** Since product of two non-zero real numbers is again a non-zero a real number.

i.e., $\forall\ a, b \in R, a\ .\ b \in R.$

(ii) **Associative axiom** **:** Multiplication is always associative in $R - \{0\}$

i.e., $a\ .\ (b\ .\ c) = (a\ .\ b)\ .\ c\ \ \forall\ a, b, c \in R - \{0\}$

$\therefore$ associative axiom is true.

(iii) **Identity axiom** **:** The identity element is $1 \in \mathrm{R} - \{0\}$ under multiplication and

$1\ .\ a = a\ .\ 1 = a, \forall\ a \in R - \{0\}$

$\therefore$ Identity axiom is true.

(iv) **Inverse axiom** **:** $\forall\ a \in R - \{0\}, \dfrac{1}{a} \in R - \{0\}$ such that

$a\ .\ \dfrac{1}{a} = \dfrac{1}{a}\ .\ a = 1$ (identity element). $\therefore$ Inverse axiom is true. $\therefore$ $(R - \{0\}, .)$ is a group.

(v) $\forall\ a, b \in R - \{0\}, a\ .\ b = b\ .\ a$

$\therefore$ Commutative property is true. $\therefore$ $(R - \{0\}, .)$ is an abelian group.

(vi) Further $R - \{0\}$ is an infinite set, $(R - \{0\}, .)$ is an infinite abelian group.

Example 9.14 : Show that the cube roots of unity forms a finite abelian group under multiplication.

Solution: Let G = $\{1, \omega, \omega^2\}$. The Cayley's table is

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

From the table, we see that,

(i) all the entries in the table are members of G. So, the closure property is true.

(ii) multiplication is always associative.

(iii) the identity element is 1 and it satisfies the identity axiom.

(iv) The inverse of 1 is 1
The inverse of ω is ω^2
the inverse of ω^2 is ω
and it satisfies the inverse axiom also. $\therefore$ $(G, .)$ is a group.

(v) the commutative property is also true.
$\therefore$ $(G, .)$ is an abelian group.

(vi) Since G is a finite set, $(G, .)$ is a finite abelian group.

Example 9.15 : Prove that the set of all 4th roots of unity forms an abelian group under multiplication.

Solution: We know that the fourth roots of unity are $1, i, -1, -i$.

Let $G = \{1, i, -1, -i\}$. The Caylely's table is

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

From the table,

(i) the closure axiom is true.

(ii) multiplication is always associative in C and hence in G.

(iii) the identity element is $1 \in G$ and it satisfies the identity axiom.

(iv) the inverse of 1 is 1 ; i is $-i$; -1 is -1 ; and $-i$ is i. Further it satisfies the inverse axiom. hence $(G, .)$ is a group.

(v) From the table, the commutative property is also true.
$\therefore$ $(G, .)$ is an abelian group.

Example 9.16 : Prove that $(\boldsymbol{C}, +)$ is an infinite abelian group.

Solution:

(i) **Closure axiom :** Sum of two complex numbers is always a complex number.

i.e., $z_1, z_2 \in C \Rightarrow z_1 + z_2 \in C$

Closure axiom is true.

(ii) **Associative axiom :** Addition is always associative in C

i.e., $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \ \forall\ z_1, z_2, z_3 \in C$

$\therefore$ Associative axiom is true.

(iii) **Identity axiom :**

The identity element $o = o + io \in C$ and $o + z = z + o = z \ \forall\ z \in C$

$\therefore$ Identity axiom is true.

(iv) **Inverse axiom :** For every $z \in C$ there exists a unique $-z \in C$ such that

$z + (-z) = -z + z = 0$. Inverse is true. $\therefore$ $(C, +)$ is a group.

(v) **Commutative property :**

$\forall\ z_1, z_2 \in C$, $z_1 + z_2 = z_2 + z_1$

$\therefore$ the commutative property is true. Hence $(C , +)$ is an abelian group. Since C is an infinite set $(C, +)$ is an infinite abelian group.

Example 9.17 : Show that the set of all non-zero complex numbers is an abelian group under the usual multiplication of complex numbers.

Solution:

(i) **Closure axiom :** Let $G = C - \{0\}$ Product of two non-zero complex numbers is again a non-zero complex number.

$\therefore$ Closure axiom is true.

(ii) **Associative axiom :**

Multiplication is always associative.

$\therefore$ Associative property is true.

(iii) **Identity axiom :**

$1 = 1 + io \in G$, 1 is the identity element and $1.z = z\ .\ 1 = z \ \forall\ z \in G$.

$\therefore$ Identity axiom is true.

(iv) **Inverse axiom :**

Let $z = x + iy \in G$. Here $z \neq 0 \Rightarrow x$ and y are not both zero.

$\therefore x^2 + y^2 \neq 0$

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)\,(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right) \in G$$

Further $z\ .\ \frac{1}{z} = \frac{1}{z}\ .\ z = 1$ $\therefore$ z has the inverse $\frac{1}{z} \in G$.

Thus inverse axiom is satisfied. $\therefore$ $(G, \mathbf{.})$ is a group.

(v) **Commutative property :**

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$
$$= (ca - db) + i(da + cb) = z_2 z_1$$

$\therefore$ It satisfies the commutative property.

$\therefore$ G is an abelian group under the usual multiplication of complex numbers.

Note : Here the number 0 is removed, because 0 has no inverse under multiplication. We can also show that $Q - \{0\}$, $R - \{0\}$ are abelian groups under multiplication. But $Z - \{0\}$ is not a group under multiplication.

$\because$ $7 \in Z - \{0\}$ while its inverse $\frac{1}{7} \notin Z - \{0\}$

Note : While verifying the axioms, follow the order given in the definition. If one axiom fails, stop the process at that stage. There is no use in continuing further.

The following table shows which number systems are satisfying the axioms of a group in the order for a particular operation.

*	N	E	Z	Q	R	C	$Q - \{0\}$	$R - \{0\}$	$C - \{0\}$
+	Semi group	group	group	group	group	group	not closed	not closed	not closed
.	monid	semi-group	monoid	monoid	monoid	monoid	group	group	group
−	not closed	not associative	not associative	not associative	not associative	not associative	not closed	not closed	not closed
÷	not closed	not closed	not closed	not closed	not closed	not closed	not associative	not associative	not associative

Example 9.18 : Show that $(Z, *)$ is an infinite abelian group where * is defined as $a * b = a + b + 2$.

Solution:

(i) **Closure axiom :** Since a, b and 2 are integers $a + b + 2$ is also an integer.

$\therefore$ $a * b \in z \quad \forall\, a, b \in z$

Thus closure axiom is true.

(ii) **Associative axiom :**

Let $a, b, c \in G$

$$(a * b) * c = (a + b + 2) * c = (a + b + 2) + c + 2 = a + b + c + 4$$
$$a * (b * c) = a * (b + c + 2) = a + (b + c + 2) + 2 = a + b + c + 4$$
$$\Rightarrow \quad (a * b) * c = a * (b * c)$$

Thus associative axiom is true.

(iii) **Identity axiom :**

Let e be the identity element.

By the definition of e, $a * e = a$

By the definition of $*$, $a * e = a + e + 2$

$\Rightarrow a + e + 2 = a$

$\Rightarrow e = -2$

$-2 \in Z$. Thus identity axiom is true.

(iv) **Inverse axiom :**

Let $a \in G$ and a^{-1} be the inverse element of a

By the definition of a^{-1}, $a * a^{-1} = e = -2$

By the definition of $*$, $a * a^{-1} = a + a^{-1} + 2$

$\Rightarrow a + a^{-1} + 2 = -2$

$\Rightarrow a^{-1} = -a - 4$

Clearly $-a-4 \in Z$. $\therefore$ Inverse axiom is true. $\therefore$ $(Z, *)$ is a group.

(v) **Commutative property :**

Let $a, b \in G$

$a * b = a + b + 2 = b + a + 2 = b * a$ $\therefore$ $*$ is commutative.

$\therefore$ $(Z, *)$ is an abelian group. further, Z is an infinite set. The group is an infinite abelian group.

Example 9.19 : Show that the set of all 2×2 non-singular matrices forms a non-abelian infinite group under matrix multiplication, (where the entries belong to R).

Solution:

Let G be the set of all 2×2 non-singular matrices, where the entries belong to R.

(i) **Closure axiom :** Since product of two non-singular matrices is again non-singular and the order is 2×2, the closure axiom is satisfied.

i.e., $A, B \in G \Rightarrow AB \in G$.

(ii) **Associative axiom :** Matrix multiplication is always associative and hence associative axiom is true. i.e., $A(BC) = (AB)C \ \forall \ A, B, C \in G$.

(iii) **Identity axiom :** The identity element is $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ and it satisfies the identity property.

(iv) **Inverse axiom :** the inverse of $A \in G$, exists i.e. A^{-1} exists and is of order 2×2 and $AA^{-1} = A^{-1}A = I$. Thus the inverse axiom is satisfied. Hence the set of all 2×2 non-singular matrices forms a group under matrix multiplication. Further, matrix multiplication is non-commutative (in general) and the set contain infinitely many elements. The group is an infinite non-abelian group.

Example 9.20 : Show that the set of four matrices

$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$ form an abelian group, under multiplication of matrices.

Solution:

Let $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, $A = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$, $B = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, $C = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$ and let $G = \{I, A, B, C\}$

By computing the products of these matrices, taken in pairs, we can form the multiplication table as given below :

.	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

(i) All the entries in the multiplication tables are members of G. So, G is closed under **.** $\therefore$ Closure axiom is true.

(ii) Matrix multiplication is always associative

(iii) Since the row headed by I coincides with the top row and the column headed by I coincides with the extreme left column, I is the identity element in G.

(iv) $I . I = I \Rightarrow I$ is the inverse of I

$A . A = I \Rightarrow A$ is the inverse of A

$B . B = I \Rightarrow B$ is the inverse of B

$C . C = I \Rightarrow C$ is the inverse of C

From the table it is clear that **.** is commutative. $\therefore$ G is an abelian group under matrix multiplication.

Example 9.21 : Show that the set G of all matrices of the form $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$, where $x \in R - \{0\}$, is a group under matrix multiplication.

Solution:

Let $G = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} / x \in R - \{0\} \right\}$ we shall show that G is a group under matrix multiplication.

(i) **Closure axiom :**

$$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in G, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in G$$

$$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in G, \ (\because x \neq 0, \ y \neq 0 \ \Rightarrow \ 2xy \neq 0)$$

i.e., G is closed under matrix multiplication.

(ii) Matrix multiplication is always associative.

(iii) Let $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix} \in G$ be such that $AE = A$ for every $A \in G$.

$$AE = A \Rightarrow \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \Rightarrow 2xe = x \Rightarrow e = \frac{1}{2} \ (\because x \neq 0)$$

Thus $E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in G$ is such that $AE = A$, for every $A \in G$

We can similarly show that $EA = A$ for every $A \in G$.

$\therefore$ E is the identity element in G and hence identity axiom is true.

(iv) Suppose $A^{-1} = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in G$ is such that $A^{-1}A = E$

Then we have $\begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \Rightarrow 2xy = \frac{1}{2} \Rightarrow y = \frac{1}{4x}$

$\therefore A^{-1} = \begin{bmatrix} 1/4\,x & 1/4\,x \\ 1/4\,x & 1/4\,x \end{bmatrix} \in G$ is such that $A^{-1}A = E$

Similarly we can show that $A\,A^{-1} = E$. $\therefore$ A^{-1} is the inverse of A.

$\therefore$ G is a group under matrix multiplication.

Note : The above group is abelian since $AB = BA$. But in general matrix multiplication is not commutative.

Example 9.22 : Show that the set $G = \left\{a + b\sqrt{2} \;/\; a, b \in Q\right\}$ is an infinite abelian group with respect to addition.

Solution:

(i) **Closure axiom :**

Let $x, y \in G$. Then $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$; $a, b, c, d \in Q$.

$x + y = \left(a + b\sqrt{2}\right) + \left(c + d\sqrt{2}\right) = (a + c) + (b + d)\sqrt{2} \in G$,

since $(a + c)$ and $(b + d)$ are rational numbers.

$\therefore$ G is closed with respect to addition.

(ii) **Associative axiom :** Since the elements of G are all real numbers, addition is associative.

(iii) **Identity axiom :**

There exists $0 = 0 + 0\sqrt{2} \in G$ such that for all $x = a + b\sqrt{2} \in G$,

$$\begin{aligned} x + 0 &= \left(a + b\sqrt{2}\right) + \left(0 + 0\sqrt{2}\right) \\ &= a + b\sqrt{2} = x \end{aligned}$$

Similarly, we have $0 + x = x$. $\therefore$ 0 is the identity element of G and satisfies the identity axiom.

(iv) **Inverse axiom :**

For each $x = a + b\sqrt{2} \in G$, there exists $-x = (-a) + (-b)\sqrt{2} \in G$

such that $$\begin{aligned} x + (-x) &= \left(a + b\sqrt{2}\right) + \left((-a) + (-b)\sqrt{2}\right) \\ &= \left(a + (-a)\right) + \left(b + (-b)\right)\sqrt{2} = 0 \end{aligned}$$

Similarly we have $(-x) + x = 0$

$\therefore$ $(-a) + (-b)\sqrt{2}$ is the inverse of $a + b\sqrt{2}$ and satisfies the inverse axiom. $\therefore$ G is a group under addition.

(v) **Commutative axiom :**

$$\begin{aligned} x + y &= (a + c) + (b + d)\sqrt{2} = (c + a) + (d + b)\sqrt{2} \\ &= \left(c + d\sqrt{2}\right) + \left(a + b\sqrt{2}\right) \\ &= y + x, \text{ for all } x, y \in G. \end{aligned}$$ $\therefore$ The commutative property is true.

$\therefore$ $(G, +)$ is an abelian group. Since G is infinite, we see that $(G, +)$ is an infinite abelian group.

Example 9.23 : Let G be the set of all rational numbers except 1 and * be defined on G by $a * b = a + b - ab$ for all $a, b \in G$. Show that $(G, *)$ is an infinite abelian group.

Solution: Let $G = Q - \{1\}$

Let $a, b \in G$. Then a and b are rational numbers and $a \neq 1$, $b \neq 1$.

(i) **Closure axiom :** Clearly $a * b = a + b - ab$ is a rational number. But to prove $a * b \in G$, we have to prove that $a * b \neq 1$.

On the contrary, assume that $a * b = 1$ then

$$\begin{aligned} a + b - ab &= 1 \\ \Rightarrow b - ab &= 1 - a \\ \Rightarrow b(1 - a) &= 1 - a \\ \Rightarrow b &= 1 \quad (\because a \neq 1,\ 1 - a \neq 0) \end{aligned}$$

This is impossible, because $b \neq 1$. $\therefore$ Our assumption is wrong.

$\therefore$ $a * b \neq 1$ and hence $a * b \in G$.

$\therefore$ Closure axiom is true.

(ii) **Associative axiom :**

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a\,(b + c - bc) \\ &= a + b + c - bc - ab - ac + abc \\ (a * b) * c &= (a + b - ab) * c \\ &= (a + b - ab) + c - (a + b - ab)\,c \\ &= a + b + c - ab - ac - bc + abc \\ \therefore\ a * (b * c) &= (a * b) * c \ \forall\ a, b, c \in G \end{aligned}$$

$\therefore$ Associative axiom is true.

(iii) **Identity axiom :** Let e be the identity element.

By definition of e, $a * e = a$

By definition of $*$, $a * e = a + e - ae$

$$\begin{aligned} \Rightarrow a + e - ae &= a \\ \Rightarrow e(1 - a) &= 0 \\ \Rightarrow e &= 0 \text{ since } a \neq 1 \\ e &= 0 \in G \end{aligned}$$

$\therefore$ Identity axiom is satisfied.

(iv) **Inverse axiom :**

Let a^{-1} be the inverse of $a \in G$.

By the definition of inverse, $a * a^{-1} = e = 0$

By the definition of $*$, $a * a^{-1} = a + a^{-1} - aa^{-1}$

$$\begin{aligned} \Rightarrow a + a^{-1} - aa^{-1} &= 0 \\ \Rightarrow a^{-1}(1 - a) &= -a \\ \Rightarrow a^{-1} &= \frac{a}{a - 1} \in G \text{ since } a \neq 1 \end{aligned}$$

$\therefore$ Inverse axiom is satisfied. $\therefore$ $(G, *)$ is a group.

(v) **Commutative axiom :**

For any $a, b \in G$, $$a * b = a + b - ab = b + a - ba = b * a$$

$\therefore$ * is commutative in G and hence $(G, *)$ is an abelian group. Since G is infinite, $(G, *)$ is an infinite abelian group.

Example 9.24 : Prove that the set of four functions f_1, f_2, f_3, f_4 on the set of non-zero complex numbers $\boldsymbol{C} - \{0\}$ defined by

$f_1(z) = z,\ f_2(z) = -z,\ f_3(z) = \dfrac{1}{z}$ and $f_4(z) = -\dfrac{1}{z}\ \forall\ z \in \boldsymbol{C} - \{0\}$ forms an abelian group with respect to the composition of functions.

Solution: Let $G = \{f_1, f_2, f_3, f_4\}$

$$(f_1 \circ f_1)(z) = f_1(f_1(z)) = f_1(z)$$

$$\therefore f_1 \circ f_1 = f_1$$

$$f_2 \circ f_1 = f_2,\ f_3 \circ f_1 = f_3,\ f_4 \circ f_1 = f_4$$

Again $$(f_2 \circ f_2)(z) = f_2(f_2(z)) = f_2(-z) = -(-z) = z = f_1(z)$$

$$\therefore f_2 \circ f_2 = f_1$$

Similarly $$f_2 \circ f_3 = f_4,\ f_2 \circ f_4 = f_3$$

$$(f_3 \circ f_2)(z) = f_3(f_2(z)) = f_3(-z) = -\frac{1}{z} = f_4(z)$$

$$\therefore f_3 \circ f_2 = f_4$$

Similarly $$f_3 \circ f_3 = f_1,\ f_3 \circ f_4 = f_2$$

$$(f_4 \circ f_2)(z) = f_4(f_2(z)) = f_4(-z) = -\frac{1}{-z} = \frac{1}{z} = f_3(z)$$

$$\therefore f_4 \circ f_2 = f_3$$

Similarly $$f_4 \circ f_3 = f_2,\ f_4 \circ f_4 = f_1$$

Using these results we have the composition table as given below :

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

From the table

(i) All the entries of the composition table are the elements of G .

$\therefore$ Closure axiom is true.

(ii) Composition of functions is in general associative.

(iii) Clearly f_1 is the identity element of G and satisfies the identity axiom.

(iv) From the table :

Inverse of f_1 is f_1 ; Inverse of f_2 is f_2

Inverse of f_3 is f_3 ; Inverse of f_4 is f_4

Inverse axiom is satisfied. (G, o) is a group.

(v) From the table the commutative property is also true.

$\therefore$ (G, o) is an abelian group.

9.2.3 Modulo Operation

We shall now define new types of operations called "Addition modulo n" and "Multiplication modulo n", where n is a positive integer. To define these operations we require the notion of "Division Algorithm".

Let $a, b \in Z$ with $b \neq 0$. Then we can divide a by b to get a quotient q and a non-negative remainder r which is smaller in size than b.

i.e., $a = qb + r$, where $0 \leq r < |b|$. This is called "Division Algorithm".

For example, if $a = 17$, $b = 5$ then $17 = (3 \times 5) + 2$

Here $q = 3$ and $r = 2$

Addition modulo n ($+_n$) :

Let $a, b \in Z$ and n be a fixed positive integer. We define addition modulo n by $a +_n b = r$; $0 \leq r < n$ where r is the least non-negative remainder when $a + b$ is divided by n.

For example, if $a = 25$, $b = 8$ and $n = 7$ then $25 +_7 8 = 5$

($\because 25 + 8 = 33 = (4 \times 7) + 5$)

Multiplication modulo n ($\cdot_n$)

As given above

$a \cdot_n b = r$; $0 \leq r < n$, where r is the least non-negative remainder when ab is divided by n.

For example, $2 \cdot_5 4 = 3$

$7 \cdot_9 8 = 2$

Congruence modulo n :

Let $a, b \in Z$ and n be a fixed positive integer.

We say that "a is congruent to b modulo n" $\Leftrightarrow$ $(a - b)$ is divisible by n

Symbolically,

$a \equiv b \pmod{n} \Leftrightarrow (a - b)$ is divisible by n.

$15 \equiv 3 \pmod{4}$ is true because $15 - 3$ is divisible by 4.

$17 \equiv 4 \pmod{3}$ is not true because $17 - 4$ is not divisible by 3.

Congruence classes modulo n :

Let $a \in Z$ and n be a fixed positive integer.

Collect all numbers which are congruent to 'a' modulo n. This set will be denoted as $[a]$ and is called the congruence class modulo n or residue class modulo n.

Thus
$$\begin{aligned}[a] &= \{x \in Z \,/\, x \equiv a \pmod{n}\}\\ &= \{x \in Z \,/\, (x - a) \text{ is divisible by } n\}\\ &= \{x \in Z \,/\, (x - a) \text{ is a multiple of } n\}\\ &= \{x \in Z \,/\, (x - a) = kn\},\ k \in Z\\ &= \{x \in Z \,/\, x = a + kn\},\ k \in Z\end{aligned}$$

consider the congruence classes modulo 5.

$$\begin{aligned}[a] &= \{x \in Z \,/\, x = a + kn\}\\ [0] &= \{x \in Z \,/\, x = 5k,\ k \in Z\} = \{\ldots -10, -5, 0, 5, 10 \ldots\}\\ [1] &= \{x \in Z \,/\, x = 5k + 1,\ k \in Z\} = \{\ldots -9, -4, 1, 6, 11, \ldots\}\\ [2] &= \{x \in Z \,/\, x = 5k + 2,\ k \in Z\} = \{\ldots -8, -3, 2, 7, 12, \ldots\}\\ [3] &= \{x \in Z \,/\, x = 5k + 3,\ k \in Z\} = \{\ldots -7, -2, 3, 8, 13, \ldots\}\\ [4] &= \{x \in Z \,/\, x = 5k + 4,\ k \in Z\} = \{\ldots -6, -1, 4, 9, 14 \ldots\}\\ [5] &= \{x \in Z \,/\, x = 5k + 5,\ k \in Z\} = \{\ldots -5, 0, 5, 10, \ldots\} = [0]\end{aligned}$$

Similarly $[6] = [1]$; $[7] = [2]$; etc.

Note that, we have only 5 distinct classes whose union gives the entire Z.

Thus the set of congruence classes corresponding to 5 is $\{[0], [1], [2], [3], [4]\}$ and it will be deonoted by Z_5.

i.e., $Z_5 = \{[0], [1], [2], [3], [4]\}$

If we take the modulo 6, we have $Z_6 = \{[0], [1] \ldots. [5]\}$.

Thus for any positive integer n, we have $Z_n = \{[0], [1] \ldots [n - 1]\}$

Here $[n] = [0]$ and the union of these classes gives Z.

Operations on congruence classes :

(1) Addition :

Let $[a], [b] \in Z_n$

$$[a] +_n [b] = [a+b] \text{ if } a+b < n$$
$$= [r] \text{ if } a+b \geq n$$

Where r is the least non-negative remainder when $a + b$ is divided by n.

For example,

In Z_{10}, $[5] +_{10} [7] = [2]$

In Z_8, $[3] +_8 [5] = [0]$

(ii) Multiplication :

$$[a] \cdot_n [b] = \begin{cases} [ab] & \text{if } ab < n \\ [r] & \text{if } ab \geq n \end{cases}$$

where r is the least non-negative remainder when ab is divided by n

In Z_5 $\quad [2] \cdot_5 [2] = [4]$

$\quad [3] \cdot_5 [4] = 2$

In Z_7, $\quad [3] \cdot_7 [3] = [2]$

In Z_8, $\quad [5] \cdot_8 [3] = [7]$

Example 9.25 : Show that $(Z_n, +_n)$ forms group.

Solution: Let $Z_n = \{[0], [1], [2], ... [n-1]\}$ be the set of all congruence classes modulo n. and let $[l], [m], \in Z_n \quad 0 \leq l, m, < n$

(i) **Closure axiom :** By definition

$$[l] +_n [m] = \begin{cases} [l+m] & \text{if } l+m < n \\ [r] & \text{if } l+m \geq n \end{cases}$$

where $l + m = q \,.\, n + r \quad 0 \leq r < n$

In both the cases, $[l + m] \in Z_n$ and $[r] \in Z_n$

$\therefore$ Closure axiom is true.

(ii) Addition modulo n is always associative in the set of congruence classes modulo n.

(iii) The identity element $[0] \in Z_n$ and it satisfies the identity axiom.

(iv) The inverse of $[l] \in Z_n$ is $[n - l]$

Clearly $[n - l] \in Z_n$ and

$$[l] +_n [n - l] = [0]$$
$$[n - l] +_n [l] = [0]$$

$\therefore$ The inverse axiom is also true. Hence $(Z_n, +_n)$ is a group.

Note : $(Z_n, +_n)$ is a finite abelian group of order n.

Example 9.26 : Show that $(Z_7 - \{[0]\}, \cdot_7)$ forms a group.

Solution: Let $G = [[1], [2], ... [6]]$

The Cayley's table is

$\cdot_7$	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

From the table :

(i) all the elements of the composition table are the elements of G. $\therefore$ The closure axiom is true.

(ii) multiplication modulo 7 is always associative.

(iii) the identity element is $[1] \in G$ and satisfies the identity axiom.

(iv) the inverse of [1] is [1] ; [2] is [4] ; [3] is [5] ; [4] is [2] ; [5] is [3] and [6] is [6] and it satisfies the inverse axiom.

$\therefore$ the given set forms a group under multiplication modulo 7.

In general, it can be shown that $(Z_p - \{(0)\}, \cdot_p)$ is a group for any prime p. But the proof is beyond the scope of this book.

Note : Does the set of all non-zero congruence classes modulo n, a positive integer, form a group under multiplication modulo n, ?

Example 9.27 : Show that the nth roots of unity form an abelian group of finite order with usual multiplication.

Solution: We know that $1, \omega, \omega^2 ...... \omega^{n-1}$ are the n^{th} roots of unity, where $\omega = \text{cis}\,\frac{2\pi}{n}$. Let $G = \{1, \omega, \omega^2 ... \omega^{n-1}\}$

(i) **Closure axiom :** Let $\omega^l, \omega^m \in G$, $0 \le l, m \le (n - 1)$

To prove $\omega^l \omega^m = \omega^{l+m} \in G$

Case (i) $l + m < n$

If $l + m < n$ then clearly $\omega^{l+m} \in G$

Case (ii) $l + m \geq n$ By division algoritham,

$l + m = (q \,.\, n) + r$ where $0 \leq r < n$, q is a positive integer.

$\omega^{l+m} = \omega^{qn+r} = (\omega^n)^q \,.\, \omega^r = (1)^q \omega^r = \omega^r \in G \quad \because 0 \leq r < n$

Closure property is true.

(ii) **Associative axiom :** Multiplication is always associative in the set of complex numbers and hence in G

$$\omega^l .(\omega^p.\omega^m) = \omega^l \,.\, \omega^{(p+m)} = \omega^{l+(p+m)} = \omega^{(l+p)+m} = (\omega^{l+p}) \,.\, \omega^m$$
$$= (\omega^l \,.\, \omega^p) \,.\, \omega^m = \forall\ \omega^l, \omega^m, \omega^p \in \mathrm{G}$$

(iii) **Identity axiom :** The identity element $1 \in \mathrm{G}$ and it satisfies

$1.\omega^l = \omega^l .1 = \omega^l \ \forall\ \omega^l \in G$

(iv) **Inverse axiom :**

For any $\omega^l \in G$, $\omega^{n-l} \in G$ and $\omega^l \,.\, \omega^{n-l} = \omega^{n-l} .\omega^l = \omega^n = 1$

Thus inverse axiom is true.

$\therefore$ $(G, \mathbf{.})$ is a group.

(v) **Commutative axiom :**

$\omega^l \,.\, \omega^m = \omega^{l+m} = \omega^{m+l} = \omega^m \,.\, \omega^l \quad \forall\ \omega^l, \omega^m \in G$

$\therefore$ $(G, \mathbf{.})$ is an abelian group. Since G contains n elements, $(G, \mathbf{.})$ is a finite abelian group of order n.

9.2.4 Order of an element :

Let G be a group and $a \in G$. The order of 'a' is defined as the least positive integer n such that $a^n = e$, e is the identity element. If no such positive integer exists, then a is said to be of infinite order. The order of a is denoted by $0(a)$.

Note : Here $a^n = a * a * a \ldots *a$ (n times). If * is usual multiplication '**.**' then a^n is $a \,.\, a \,.a\ldots$ (n times) i.e., a^n .

If * is usual addition then a^n is $a + a + a + \ldots + a$ (n times) i.e., na. Thus a^n is not "a to the power n", it is a symbol to denote $a * a * a \ldots * a$ (n times). Clearly $a^n \in G$, if $a \in G$. (By the repeated application of closure axiom).

Theorem :

For any group G, the identity element is the only element of order 1.

Proof : If a ($\neq e$) is another element of order 1 then by the definition of order of an element, we have $(a)^1 = e \Rightarrow a = e$ which is a contradiction. $\therefore$ e is the only element of order 1.

Example 9.28 : Find the order of each element of the group (G, **.**) where $G = \{1, -1, i, -i\}$.

Solution: In the given group, the identity element is 1. $\therefore$ 0(1) = 1.

0(– 1) = 2 [$\because$ we have to multiply – 1 two times (minimum) to get 1 i.e., (– 1) (– 1) = 1]

0(i) = 4 [$\because$ we have to multiply i four times to get 1, i.e., (i) (i) (i) (i) = 1]

0(– i) = 4 [$\because$ we have to multiply – i four times to get 1].

Example 9.29 : Find the order of each element in the group $G = \{1, \omega, \omega^2\}$, consisting of cube roots of unity with usual multiplication.

Solution: We know that the identity element is 1. $\therefore$ 0(1) = 1.

$$0(\omega) = 3. \quad \text{Since } \omega \,.\, \omega \,.\, \omega = \omega^3 = 1$$

$$0(\omega^2) = 3 \text{ since } (\omega^2)\,(\omega^2)\,(\omega^2) = \omega^6 = 1$$

Example 9.30 : Find the order of each element of the group (Z_4, $+_4$)

Solution: $Z_4 = \{[0], [1], [2], [3]\}$ is an abelian group under the addition modulo 4. The identity element is [0] and note that [4] = [8] = [12] = [0]

$\therefore$ 0([0]) = 1

0([1]) = 4 [$\because$ we have to add [1] four times to get [4] or [0]]

0 ([2]) = 2 [$\because$ we have to add [2] two times to get [4] or [0]]

0 ([3]) = 4 $\because$ we have to add [3] four times to get [12] or [0]

9.2.5 Properties of Groups :

Theorem :

The identity element of a group is unique.

Proof : Let G be a group. If possible let e_1 and e_2 be identity elements in G.

Treating e_1 as an identity element we have $e_1 * e_2 = e_2$ … (1)

Treating e_2 as an identity element, we have $e_1 * e_2 = e_1$ … (2)

From (1) and (2), $e_1 = e_2$

$\therefore$ Identity element of a group is unique.

Theorem :

The inverse of each element of a group is unique.

Proof :

Let G be a group and let $a \in G$.

If possible, let a_1 and a_2 be two inverses of a.

Treating a_1 as an inverse of 'a' we have $a * a_1 = a_1 * a = e$.

Treating a_2 as an inverse of 'a', we have $a * a_2 = a_2 * a = e$

Now $\quad a_1 = a_1 * e = a_1 * (a * a_2) = (a_1 * a) * a_2 = e * a_2 = a_2$

$\Rightarrow$ Inverse of an element is unique.

Theorem : (Cancellation laws)

Let G be a group. Then for all $a, b, c \in G$,

(i) $a * b = a * c \Rightarrow b = c$ (Left Cancellation Law)

(ii) $b * a = c * a \Rightarrow b = c$ (Right Cancellation Law)

Proof : (i)

$$\begin{aligned} a * b = a * c &\Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c) \\ &\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c \\ &\Rightarrow e * b = e * c \\ &\Rightarrow b = c \end{aligned}$$

(ii)

$$\begin{aligned} b * a = c * a &\Rightarrow (b * a) * a^{-1} = (c * a) * a^{-1} \\ &\Rightarrow b * (a * a^{-1}) = c * (a * a^{-1}) \\ &\Rightarrow b * e = c * e \\ &\Rightarrow b = c \end{aligned}$$

Theorem : In a group G, $(a^{-1})^{-1} = a$ for every $a \in G$.

Proof :

We know that $a^{-1} \in G$ and hence $(a^{-1})^{-1} \in G$. Clearly $a * a^{-1} = a^{-1} * a = e$

$$a^{-1} * (a^{-1})^{-1} = (a^{-1})^{-1} * a^{-1} = e$$

$$\Rightarrow a * a^{-1} = (a^{-1})^{-1} * a^{-1}$$

$$\Rightarrow a = (a^{-1})^{-1} \quad \text{(by Right Cancellation Law)}$$

Theorem : (Reversal law)

Let G be a group $a, b \in G$. Then $(a * b)^{-1} = b^{-1} * a^{-1}$.

Proof : It is enough to prove $b^{-1} * a^{-1}$ is the inverse of $(a * b)$

$\therefore$ To prove (i) $(a * b) * (b^{-1} * a^{-1}) = e$

(ii) $(b^{-1} * a^{-1}) * (a * b) = e$

(i)

$$\begin{aligned} (a * b) * (b^{-1} * a^{-1}) &= a * (b * b^{-1}) * a^{-1} \\ &= a * (e) * a^{-1} \\ &= a * a^{-1} = e \end{aligned}$$

(ii) $(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b$

$= b^{-1} * (e) * b$

$= b^{-1} * b = e$

$\therefore\ b^{-1} * a^{-1}$ is the inverse of $a * b$ i.e., $(a * b)^{-1} = b^{-1} * a^{-1}$

EXERCISE 9.4

(1) Let S be a non-empty set and o be a binary operation on S defined by $xoy = x$; $x, y \in S$. Determine whether o is commutative and associative.

(2) Show that the set N of natural members is a semi-group under the operation $x * y = \max\{x, y\}$. Is it a monoid?

(3) Show that the set of all positive even integers forms a semi-group under the usual addition and multiplication. Is it a monoid under each of the above operations?

(4) Prove that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ form a group under matrix multiplication.

(5) Show that the set G of all positive rationals forms a group under the composition * defined by $a * b = \dfrac{ab}{3}$ for all $a, b \in G$.

(6) Show that $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}\right\}$ where $\omega^3 = 1$, $\omega \neq 1$ form a group with respect to matrix multiplication.

(7) Show that the set M of complex numbers z with the condition $|z| = 1$ forms a group with respect to the operation of multiplication of complex numbers.

(8) Show that the set G of all rational numbers except -1 forms an abelian group with respect to the operation * given by $a * b = a + b + ab$ for all $a, b \in G$.

(9) Show that the set $\{[1], [3], [4], [5], [9]\}$ forms an abelian group under multiplication modulo 11.

(10) Find the order of each element in the group $\left(Z_5 - \{[0]\}, \cdot_5\right)$

(11) Show that the set of all matrices of the form $\begin{pmatrix} a & o \\ o & o \end{pmatrix}$, $a \in R - \{0\}$ forms an abelian group under matrix multiplication.

(12) Show that the set $G = \{2^n / n \in Z\}$ is an abelian group under multiplication.

10. PROBABILITY DISTRIBUTIONS

10.1 Introduction :

In XI Standard we dealt with random experiments which can be described by finite sample space. We studied the assignment and computation of probabilities of events. In the Sciences one often deals with variables as a ‘quantity that may assume any one of a set of values’. In Statistics we deal with random variables - variables whose observed value is determined by chance.

10.2. Random Variable :

The outcomes of an experiment are represented by a random variable if these outcomes are numerical or if real numbers can be assigned to them.

For example, in a die rolling experiment, the corresponding random variable is represented by the set of outcomes {1, 2, 3, 4, 5, 6} ; while in the coin tossing experiment the outcomes head *(H)* or tail *(T)* can be represented as a random variable by assuming 0 to *T* and 1 to *H*. In this sense a random variable is a real valued function that maps the sample space into the real line.

Let us consider the tossing of two fair coins at a time. The possible results are {*HH*, *TH*, *HT*, *TT*}. Let us consider the variable X which is “the number of heads obtained” while tossing two fair coins. We could assign the value $X = 0$ to the outcome of getting no heads, $X = 1$ to the outcome of getting only 1 head and $X = 2$ to the out come of getting 2 heads.

Therefore $X(TT) = 0$, $X(TH) = 1$, $X(HT) = 1$ and $X(HH) = 2$. Therefore X takes the values 0,1,2. Thus we can assign a real number $X(s)$ to every element s of the sample space S.

Definition : If S is a sample space with a probability measure and X is a real valued function defined over the elements of S, then X is called a random variable.

A random variable is also called a chance variable or a stochastic variable.

Types of Random variables :

(1) Discrete Random variable (2) Continuous Random variable

10.2.1 Discrete Random Variable :

Definition : Discrete Random Variable

If a random variable takes only a finite or a countable number of values, it is called a discrete random variable.

Note : Biased coins may have both sides marked as tails or both sides marked as heads or may fall on one side only for every toss, whereas a fair or unbiased coin means, it has equal chances of falling on heads and tails. Similarly biased dice may have repeated numbers on several sides ; some numbers may be missing. For a fair die the probability of getting any number from one to six will be 1/6.

Example :

1. The number of heads obtained when two coins are tossed is a discrete random variable as X assumes the values 0, 1 or 2 which form a countable set.
2. Number of Aces when ten cards are drawn from a well shuffled pack of 52 cards.

The random variable X assumes 0, 1, 2, 3 or 4 which is again a countable set.

i.e., X (No aces) = 0, X (one ace) = 1, X (two aces) = 2,

X (three aces) = 3, X (four aces) = 4

Probability Mass Function :

The Mathematical definition of discrete probability function $p(x)$ is a function that satisfies the following properties :

(1) The probability that X can take a specific value x is $p(x)$ ie., $P(X = x) = p(x) = p_x$.

(2) $p(x)$ is non – negative for all real x.

(3) The sum of $p(x)$ over all possible values of X is one. That is $\sum p_i = 1$ where j represents all possible values that X can have and p_i is the probability at $X = x_i$

If $a_1, a_2, \ldots a_m, a, b_1, b_2, \ldots b_n, b$ be the values of the discrete random variable X in ascending order then

(i) $P(X \geq a) = 1 - P(X < a)$

(ii) $P(X \leq a) = 1 - P(X > a)$

(iii) $P(a \leq X \leq b) = P(X = a) + P(X = b_1) + P(X = b_2) + \ldots$

$$\ldots + P(X = b_n) + P(X = b).$$

Distribution function : (Cumulative Distribution function)

The distribution function of a random variable X is defined as $F(x) = P(X \leq x) = \sum_{x_i \leq x} p(x_i) : (-\infty < x < \infty)$.

Properities of Distribution function :

1) $F(x)$ is a non-decreasing function of x

2) $0 \leq F(x) \leq 1, -\infty < x < \infty$

3) $F(-\infty) = \underset{x \to -\infty}{\mathrm{Lt}} F(x) = 0$

4) $F(\infty) = \underset{x \to +\infty}{Lt} F(x) = 1$

5) $P(X = x_n) = F(x_n) - F(x_{n-1})$

Illustration :

Find the probability mass function and cumulative distribution function for getting number of heads when three coins are tossed once.

Solution : Let X be the random variable "getting number of Heads". Sample space when three coins are tossed is

S	=	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
↓		↓	↓	↓	↓	↓	↓	↓	↓
R *(No.of Heads)*	:	3	2	2	2	1	1	1	0

Since X is the random variable getting the number of heads, X takes the values 0, 1,2 and 3. ($X : S \to R$).

P (getting no head) $= P(X = 0) = \frac{1}{8}$

P (getting one head) $= P(X = 1) = \frac{3}{8}$

P (getting two heads) $= P(X = 2) = \frac{3}{8}$

P (getting three heads) $= P(X = 3) = \frac{1}{8}$

$\therefore$ probability mass function is given by

$$P(X = x) = \begin{cases} 1/8 & \text{if } x = 0 \\ 3/8 & \text{if } x = 1 \\ 3/8 & \text{if } x = 2 \\ 1/8 & \text{if } x = 3 \end{cases} \quad \text{OR}$$

X	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

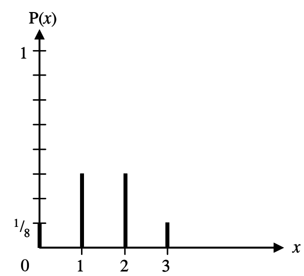


Fig. 10.1

To find cumulative distribution function.

$$\text{We have } F(x) = \sum_{x_i = -\infty}^{x} P(X = x_i)$$

When $X = 0$, $F(0) = P(X = 0) = \frac{1}{8}$

When $X = 1$, $F(1) = \sum_{i=-\infty}^{1} P(X = x_i)$

$= P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$

When $X = 2$, $F(2) = \sum_{i=-\infty}^{2} P(X = x_i)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$

When $X = 3$, $F(3) = \sum_{i=-\infty}^{3} P(X = x_i)$

$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

Cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ 1/8 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } 1 \le x < 2 \\ 7/8 & \text{if } 2 \le x < 3 \\ 1 & \text{if } 3 \le x < \infty \end{cases}$$

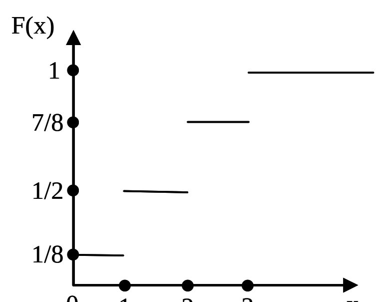


Fig. 10.2

Example 10.1 :

Find the probability mass function, and the cumulative distribution function for getting '3'*s* when two dice are thrown.

Solution :

Two dice are thrown. Let X be the random variable of getting number of '3'*s*. Therefore X can take the values 0, 1, 2.

$P(\text{no '3'}) = P(X = 0) = \dfrac{25}{36}$

$P(\text{one '3'}) = P(X = 1) = \dfrac{10}{36}$

$P(\text{two '3's}) = P(X = 2) = \dfrac{1}{36}$

Sample Space

(1,1)	(1,2)	**(1,3)**	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	**(2,3)**	(2,4)	(2,5)	(2,6)
(3,1)	**(3,2)**	$\boxed{(3,3)}$	**(3,4)**	**(3,5)**	**(3,6)**
(4,1)	(4,2)	**(4,3)**	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	**(5,3)**	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	**(6,3)**	(6,4)	(6,5)	(6,6)

probability mass function is given by

x	0	1	2
$P(X = x)$	25/36	10/36	1/36

Cumulative distribution function :

We have $F(x) = \sum_{x_i = -\infty}^{x} P(X = x_i)$

$$F(0) = P(X = 0) = \frac{25}{36}$$

$$F(1) = P(X = 0) + P(X = 1) = \frac{25}{36} + \frac{10}{36} = \frac{35}{36}$$

$$F(2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{25}{36} + \frac{10}{36} + \frac{1}{36} = \frac{36}{36} = 1$$

x	0	1	2
$F(x)$	25/36	35/36	1

Example 10.2 A random variable X has the following probability mass function

x	0	1	2	3	4	5	6
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(1) Find k.

(2) Evaluate $P(X < 4)$, $P(X \geq 5)$ and $P(3 < X \leq 6)$

(3) What is the smallest value of x for which $P(X \leq x) > \dfrac{1}{2}$.

Solution :

(1) Since P(X = x) is a probability mass function $\sum_{x=0}^{6} P(X = x) = 1$

ie., $P(X=0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = 1.$

$\Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1 \Rightarrow 49k = 1 \Rightarrow k = \dfrac{1}{49}$

(2) $P(X < 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} = \frac{16}{49}$$

$$P(X \geq 5) = P(X = 5) + P(X = 6) = \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$$

$$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{33}{49}$$

(3) The minimum value of x may be determined by trial and error method.

$$P(X \leq 0) = \frac{1}{49} < \frac{1}{2} \; ; \; P(X \leq 1) = \frac{4}{49} < \frac{1}{2}$$

$$P(X \leq 2) = \frac{9}{49} < \frac{1}{2} \; ; \; P(X \leq 3) = \frac{16}{49} < \frac{1}{2}$$

$$P(X \leq 4) = \frac{25}{49} > \frac{1}{2}$$

$\therefore$ The smallest value of x for which $P(X \leq x) > \frac{1}{2}$ is 4.

Example 10.3 : An urn contains 4 white and 3 red balls. Find the probability distribution of number of red balls in three draws one by one from the urn.

(i) with replacement (ii) without replacement

Solution : (i) with replacement

Let X be the random variable of drawing number of red balls in three draws.

$\therefore$ X can take the values 0,1,2,3.

$$P(\text{Red ball}) = \frac{3}{7} = P(R)$$

$$P(\text{Not Red ball}) = \frac{4}{7} = P(\text{w})$$

Therefore $P(X = 0) = P(\text{www}) = \frac{4}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{64}{343}$

$$P(X = 1) = P(RWW) + P(WRW) + P(WWR)$$

$$= \left(\frac{3}{7} \times \frac{4}{7} \times \frac{4}{7}\right) + \left(\frac{4}{7} \times \frac{3}{7} \times \frac{4}{7}\right) + \left(\frac{4}{7} \times \frac{4}{7} \times \frac{3}{7}\right)$$

$$= 3 \times \frac{48}{343} = \frac{144}{343}$$

$$P(X = 2) = P(RRW) + P(RWR) + P(WRR)$$

$$= \left(\frac{3}{7} \times \frac{3}{7} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{4}{7} \times \frac{3}{7}\right) + \left(\frac{4}{7} \times \frac{3}{7} \times \frac{3}{7}\right)$$

$$= 3 \times \frac{3}{7} \times \frac{3}{7} \times \frac{4}{7} = 3 \times \frac{36}{343} = \frac{108}{343}$$

$$P(X = 3) = P(RRR) = \frac{3}{7} \times \frac{3}{7} \times \frac{3}{7} = \frac{27}{343}$$

The required probability distribution is

X	0	1	2	3
$P(X = x)$	64/343	144/343	108/343	27/343

Clearly all p_i's are ≥ 0 and $\sum p_i = 1$.

2) **Without replacement :** It is also treated a simultaneous case.

Method 1 : **Using combination**	**Method 2 :** **Using Conditional Probability**
(i) P(no red ball) $P(X = 0) = \frac{4c_3 \times 3c_0}{7c_3} = \frac{4 \times 1}{35} = \frac{4}{35}$	(i) $P(www) = \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} = \frac{4}{35}$
(ii) P(1 red ball) $P(X = 1) = \frac{4c_2 \times 3c_1}{7c_3} = \frac{6 \times 3}{35} = \frac{18}{35}$	(ii) $P(Rww) + P(wRw) + P(wwR)$ $= \left(\frac{3}{7} \times \frac{4}{6} \times \frac{3}{5}\right) + \left(\frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}\right) + \left(\frac{4}{7} \times \frac{3}{6} \times \frac{3}{5}\right)$ $= 3 \times \frac{36}{210} = \frac{36}{70} = \frac{18}{35}$
(iii) P(2 red ball) $P(X = 2) = \frac{4c_1 \times 3c_2}{7c_3} = \frac{4 \times 3}{35} = \frac{12}{35}$	(iii) $P(RRw) + P(RwR) + P(wRR)$ $= \left(\frac{3}{7} \times \frac{2}{6} \times \frac{4}{5}\right) + \left(\frac{3}{7} \times \frac{4}{6} \times \frac{2}{5}\right) + \left(\frac{4}{7} \times \frac{3}{6} \times \frac{2}{5}\right)$ $= 3 \times \frac{24}{210} = \frac{12}{35}$
(iv) P(3 red ball) $P(X = 3) = \frac{4c_0 \times 3c_3}{7c_3} = \frac{1 \times 1}{35} = \frac{1}{35}$	(iv) $P(RRR) = \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} = \frac{1}{35}$

X	0	1	2	3
$P(X = x)$	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

Clearly all p_i's are ≥ 0 and $\sum p_i = 1$

10.2.2 Continuous Random Variable :

Definition : A Random Variable X is said to be continuous if it can take all possible values between certain given limits. i.e., X is said to be continuous if its values cannot be put in $1 - 1$ correspondence with N, the set of Natural numbers.

Examples for Continuous Random Variable

- The life length in hours of a certain light bulb.
- Let X denote the *ph* value of a chemical compound which is randomly selected. Then X is a continuous random variable because any *ph* value, between 0 and 14 is possible.
- If in the study of ecology of a lake, we make depth measurements at randomly chosen locations then X = the depth at such location is a continuous random variable. The limit will be between the maximum and minimum depth in the region sampled.

Probability Density Function (*p.d.f.*) :

The mathematical definition of a continuous probability function $f(x)$ is a function that satisfies the following properties.

(i) The probability that X is between two points a and b is

$$P(a \leq x \leq b) = \int_a^b f(x)\, dx$$

(ii) It is non-negative for all real X.

(iii) The integral of the probability function is 1 *i.e.,* $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Continuous probability functions are referred to as *p.d.f.*

Since continuous probability function are defined for uncountable number of points over an interval, the probability at a single point is always zero. i.e., $P(X = a) = \int_a^a f(x)\, dx = 0$.

The probabilities are measured over intervals and not at single points. That is, the area under the curve between two distinct points defines the probability for that interval.

$\therefore\ P(a \le x \le b) = P(a \le X < b) = P(a < x \le b) = P(a < x < b)$

Discrete Probability function are referred to as probability mass function and continuous probability function are referred to as probability density function. The term probability function covers both discrete and continuous distribution.

Cumulative Distribution Function :

If X is a continuous random variable, the function given by

$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$ for $-\infty < x < \infty$ where $f(t)$ is the value of the probability density function of X at t is called the distribution function or cumulative distribution of X.

Properties of Distribution function :

(i) $F(x)$ is a non-decreasing function of x

(ii) $0 \le F(x) \le 1,\ -\infty < x < \infty$.

(iii) $F(-\infty) = \underset{x \to -\infty}{lt} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{-\infty} f(x)\,dx = 0$

(iv) $F(\infty) = \underset{x \to \infty}{lt} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

(v) For any real constant a and b and $a \le b$, $P(a \le x \le b) = F(b) - \mathrm{F}(a)$

(vi) $f(x) = \dfrac{d}{dx} F(x)$

i.e., $F'(x) = f(x)$

Example 10.4 : A continuous random variable X follows the probability law,

$$f(x) = \begin{cases} k\,x\,(1-x)^{10}, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find k

Solution : Since $f(x)$ is a *p.d.f* $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_0^1 kx(1-x)^{10}\,dx = 1$

By properties of definite integral

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$$

i.e., $\int_0^1 k(1-x)\,[1-(1-x)]^{10}dx = 1$

i.e., $k\int_0^1 (1-x)x^{10}dx = 1$

i.e., $k\int_0^1 (x^{10}-x^{11})dx = 1 \Rightarrow k\left[\frac{x^{11}}{11} - \frac{x^{12}}{12}\right]_0^1 = 1 \Rightarrow k\left[\frac{1}{11}-\frac{1}{12}\right] = 1 \Rightarrow k = 132$

Example 10.5 : A continuous random variable X has p.d.f. $f(x) = 3x^2$, $0 \le x \le 1$, Find a and b such that.

(i) $P(X \le a) = P(X > a)$ and *(ii)* $P(X > b) = 0.05$

Solution :

(i) Since the total probability is 1, [Given that $P(X \le a) = P(X > a)$]

$$P(X \le a) + P(X > a) = 1$$

$$\text{i.e., } P(X \le a) + P(X \le a) = 1$$

$$\Rightarrow P(X \le a) = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2} \quad \Rightarrow \quad \int_0^a 3x^2\,dx = \frac{1}{2}$$

$$\text{i.e., } \left[\frac{3x^3}{3}\right]_0^a = \frac{1}{2} \Rightarrow a^3 = \frac{1}{2} \text{ i.e., } a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

(ii) $P(X > b) = 0.05$

$$\therefore \int_b^1 f(x)\,dx = 0.05 \quad \therefore \int_b^1 3x^2\,dx = 0.05$$

$$\left[\frac{3x^3}{3}\right]_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05$$

$$b^3 = 1 - 0.05 = 0.95 = \frac{95}{100} \Rightarrow b = \left(\frac{19}{20}\right)^{\frac{1}{3}}$$

Example 10.6 : If the probability density function of a random variable is given by $f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

find (i) k (ii) the distribution function of the random variable.

Solution: (i) Since $f(x)$ is a p.d.f. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 k(1-x^2)\,dx = 1 \quad \Rightarrow \quad k\left[x - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow k\left[1 - \frac{1}{3}\right] = 1$$

$$\Rightarrow \left(\frac{2}{3}\right)k = 1 \text{ or } k = \frac{3}{2}$$

(ii) The distribution function $F(x) = \int_{-\infty}^{x} f(t)\,dt$

(a) When $x \in (-\infty, 0]$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = 0$$

(b) When $x \in (0, 1)$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(t)\,dt = 0 + \int_0^x \frac{3}{2}(1-t^2)\,dt = \frac{3}{2}\left(x - \frac{x^3}{3}\right)$$

(c) When $x \in [1, \infty)$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = 0 + \int_0^1 \frac{3}{2}(1-t^2)\,dt + 0$$

$$= \frac{3}{2}\left[t - \frac{t^3}{3}\right]_0^1 = 1 \qquad \therefore F(x) = \begin{cases} 0 & -\infty < x \le 0 \\ 3/2\,(x - x^3/3) & 0 < x < 1 \\ 1 & 1 \le x < \infty \end{cases}$$

Example 10.7 : If $F(x) = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} x\right)$ $-\infty < x < \infty$ is a distribution function of a continuous variable X, find $P(0 \le x \le 1)$

Solution: $F(x) = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} x\right)$

$$P(0 \le x \le 1) = F(1) - F(0)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} 1\right) - \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1} 0\right)$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{4}\right] - \frac{1}{\pi}\left(\frac{\pi}{2} + 0\right) = \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{4} - \frac{\pi}{2}\right] = \frac{1}{4}$$

Example 10.8 : If $f(x) = \begin{cases} \frac{A}{x}, & 1 < x < e^3 \\ 0, & \text{elsewhere} \end{cases}$ is a probability density function of a continuous random variable X, find $p(x > e)$

Solution: Since $f(x)$ is a p.d.f. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_1^{e^3} \frac{A}{x}\,dx = 1 \Rightarrow A[\log x]_1^{e^3} = 1$$

$$\Rightarrow A[\log e^3 - \log 1] = 1 \Rightarrow A[3] = 1 \Rightarrow A = 1/3$$

Therefore $f(x) = \begin{cases} \frac{1}{3x}, & 1 < x < e^3 \\ 0 & \text{elsewhere} \end{cases}$

$$P(x > e) = \frac{1}{3}\int_e^{e^3} \frac{1}{x}\,dx = \frac{1}{3}[\log x]_e^{e^3}$$

$$= \frac{1}{3}[\log e^3 - \log e] = \frac{1}{3}[3 - 1] = \frac{2}{3}$$

Example 10.9 : For the probability density function

$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0 & , x \le 0 \end{cases}$, find $F(2)$

Solution : $F(2) = P(X \le 2) = \int_{-\infty}^{2} f(x)\,dx$

$$= \int_0^2 2\, e^{-2x}\, dx = 2 \,. \left[\frac{e^{-2x}}{-2}\right]_0^2 = -\,[e^{-4} - 1] = 1 - e^{-4} = \frac{e^4 - 1}{e^4}$$

Example 10.10 : The total life time (in year) of 5 year old dog of a certain breed is a Random Variable whose distribution function is given by

$$F(x) = \begin{cases} 0 & , \text{for } x \le 5 \\ 1 - \dfrac{25}{x^2} & , \text{for } x > 5 \end{cases}$$

Find the probability that such a five year old dog will live (i) beyond 10 years (ii) less than 8 years (iii) anywhere between 12 to 15 years.

Solution : (i) P(dog living beyond 10 years)

$$P(X > 10) = 1 - P(X \le 10)$$

$$= 1 - \left(1 - \frac{25}{x^2}\right) \quad \text{when } x = 10$$

$$= 1 - \left(1 - \frac{25}{100}\right) = 1 - \frac{3}{4} = \frac{1}{4}$$

(ii) P(dog living less than 8 years)

$$P(X < 8) = F(8) \text{ [since } P(X < 8) = P(X \le 8) \text{ for a continuous distribution]}$$

$$= \left(1 - \frac{25}{8^2}\right) = \left(1 - \frac{25}{64}\right) = \frac{39}{64}$$

(iii) P(dog living any where between 12 and 15 years) = $P(12 < x < 15)$

$$= F(15) - F(12) = \left(1 - \frac{25}{15^2}\right) - \left(1 - \frac{25}{12^2}\right) = \frac{1}{16}$$

EXERCISE 10.1

(1) Find the probability distribution of the number of sixes in throwing three dice once.

(2) Two cards are drawn successively without replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of queens.

(3) Two bad oranges are accidentally mixed with ten good ones. Three oranges are drawn at random without replacement from this lot. Obtain the probability distribution for the number of bad oranges.

(4) A discrete random variable X has the following probability distributions.

X	0	1	2	3	4	5	6	7	8
P(x)	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a (ii) Find P($x < 3$) (iii) Find P($3 < x < 7$)

(5) Verify that the following are probability density functions.

(a) $f(x) = \begin{cases} \dfrac{2x}{9}, & 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$ *(b)* $f(x) = \dfrac{1}{\pi} \dfrac{1}{(1 + x^2)}, \ -\infty < x < \infty$

(6) For the p.d.f $f(x) = \begin{cases} cx\,(1 - x)^3, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

find (i) the constant c (ii) $P\left(x < \dfrac{1}{2}\right)$

(7) The probability density function of a random variable x is

$f(x) = \begin{cases} kx^{\alpha - 1}\, e^{-\beta x^{\alpha}} & , x, \alpha, \beta > 0 \\ 0 & , \text{elsewhere} \end{cases}$. Find (i) k (ii) $P(X > 10)$

(8) For the distribution function given by $F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$

find the density function. Also evaluate

(i) $P(0.5 < X < 0.75)$ (ii) $P(X \le 0.5)$ (iii) $P(X > 0.75)$

(9) A continuous random variable x has the p.d.f defined by

$f(x) = \begin{cases} ce^{-ax}, & 0 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$. Find the value of c if $a > 0$.

(10) A random variable X has a probability density function

$f(x) = \begin{cases} k & , 0 < x < 2\pi \\ 0 & \text{elsewhere} \end{cases}$

Find (i) k (ii) $P\left(0 < X < \dfrac{\pi}{2}\right)$ (iii) $P\left(\dfrac{\pi}{2} < X < \dfrac{3\pi}{2}\right)$

10.3 Mathematical Expectation :

Expectation of a discrete random variable :

Definition : If X denotes a discrete random variable which can assume the values $x_1, x_2, \ldots\ldots\, x_n$ with respective probabilities $p_1, p_2, \ldots\, p_n$ then the mathematical expectation of X, denoted by $E(X)$ is defined by

$$E(X) = p_1 x_1 + p_2 x_2 + \ldots\ldots + p_n x_n = \sum_{i=1}^{n} p_i x_i \text{ where } \sum_{i=1}^{n} p_i = 1$$

Thus $E(X)$ is the weighted arithmetic mean of the values x_i with the weights $p(x_i)$ $\therefore$ $\overline{X} = E(X)$

Hence the mathematical Expectation $E(X)$ of a random variable is simply the arithmetic mean.

Result : If $\varphi(X)$ is a function of the random variable X,

then $E[\varphi(X)] = \sum P(X = x)\, \varphi(x)$.

Properties :

Result (1) : $E(c) = c$ where c is a constant

Proof : $E(X) = \sum p_i\, x_i$

$\therefore E(c) = \sum p_i\, c = c \sum p_i = c$ as $\sum p_i = 1$

$\therefore E(c) = c$

Result (2) : $E(cX) = c\, E(X)$

Proof : $E(cX) = \sum (cx_i)p_i = (c\, x_1)\, p_1 + (c\, x_2)\, p_2 + \ldots (c\, x_n)\, p_n$

$= c(p_1 x_1 + p_2 x_2 + \ldots\, p_n x_n)$

$= c\, E(X)$

Result (3) : $E(aX + b) = a\, E(X) + b$.

Proof : $E(aX + b) = \sum (a\, x_i + b)\, p_i$

$= (a\, x_1 + b)\, p_1 + (a\, x_2 + b) p_2 + (a\, x_n + b)\, p_n$

$= a(p_1 x_1 + p_2 x_2 + \ldots\, p_n x_n) + b \sum p_i$

$= a\, E(X) + b$. Similarly $E(aX - b) = aE(X) - b$

Moments : Expected values of a function of a random variable X is used for calculating the moments. We will discuss about two types of moments.

(i) Moments about the origin

(ii) Moments about the mean which are called central moments.

Moments about the origin :

If X is a discrete random variable for each positive integer r ($r = 1, \ldots$) the r^{th} moment

$$\boxed{\mu_r' = E(X^r) = \sum p_i\, x_i^r}$$

First moment : $\mu_1' = E(X) = \sum p_i\, x_i$

This is called the mean of the random variable X.

Second moment : $\mu_2' = E(X^2) = \sum p_i\, x_i^2$

Moments about the Mean : (Central Moments)

For each positive integer n, (n = 1, 2, ...) the n^{th} central moment of the discrete random variable X is

$$\boxed{\mu_n = E(X - \overline{X})^n = \sum(x_i - \overline{x})^n p_i}$$

First moment about the Mean $\mu_1 = E(X - \overline{X})^1 = \sum(x_i - \overline{x})^1 p_i$

$$\mu_1 = \sum x_i p_i - \overline{x} \sum p_i = \sum x_i p_i - \overline{x}\,(1) \quad \text{as} \quad \sum p_i = 1$$
$$= E(X) - E(X) = 0$$

The algebraic sum of the deviations about the arithmetic mean is always zero

2^{nd} moment about the Mean $\mu_2 = E(X - \overline{X})^2$

$$= E(X^2 + \overline{X}^2 - 2\,X\,\overline{X}) = E(X^2) + \overline{X}^2 - 2\,\overline{X}\,E(X) \quad (\because \overline{X} \text{ is a constant})$$
$$= E(X^2) + [E(X)]^2 - 2E(X)\,E(X)$$
$$\mu_2 = E(X^2) - [E(X)]^2 \quad = \mu_2' - (\mu_1')^2$$

Second moment about the Mean is called the variance of the random variable *X*

$$\boxed{\mu_2 = \text{Var}\,(X) = E(X - \overline{X})^2 = E(X^2) - [E(X)]^2}$$

Result (4) : Var $(X \pm c)$ = Var X where c is a constant.

Proof : w.k.t. Var $(X) = E(X - \overline{X})^2$

$$\text{Var}\,(X + c) = E[(X + c) - E\,(X + c)]^2$$
$$= E[X + c - E(X) - c]^2$$
$$= E[X - \overline{X}]^2 = \text{Var}\,X$$

Similarly Var $(X - c)$ = Var (X)

$\therefore$ Variance is independent of change of origin.

Result (5) : Var $(aX) = a^2$ Var (X)

Proof :
$$\text{Var}\,(aX) = E[aX - E(aX)]^2 = E[aX - aE(X)]^2$$
$$= E[a\,\{X - E(X)\}]^2$$
$$= a^2\,E[X - E(X)]^2 = a^2\,\text{Var}\,X$$

Change of scale affects the variance

Result (6) : Var (c) = 0 where c is a constant.

Proof : Var (c) = $E[c - E(c)]^2 = E[c - c]^2 = E(0) = 0$

Example 10.11 : Two unbiased dice are thrown together at random. Find the expected value of the total number of points shown up.

Solution : Let X be the random variable which represents the sum of the numbers shown in the two dice. If both show one then the sum total is 2. If both show six then the sum is 12.

The random variable X can take values from 2 to 12.

(1, 1)
(1, 2) (2, 1)
(1, 3) (2, 2) (3, 1)
(1, 4) (2, 3) (3, 2) (4, 1)
(1, 5) (2, 4) (3, 3) (4, 2) (5, 1)
(1, 6) (2, 5) (3, 4) (4, 3) (5, 2) (6, 1)
(2, 6) (3, 5) (4, 4) (5, 3) (6, 2)
(3, 6) (4, 5) (5, 4) (6, 3)
(4, 6) (5, 5) (6, 4)
(5, 6) (6, 5)
(6, 6)

$\therefore$ The probability distribution is given by.

X	2	3	4	5	6	7	8	9	10	11	12
P(X = x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(X) = \sum p_i\, x_i = \sum x_i\, p_i$$

$$= \left(2 \times \frac{1}{36}\right) + \left(3 \times \frac{2}{26}\right) + \left(4 \times \frac{3}{36}\right) + \ldots + \left(12 \times \frac{1}{36}\right) = \frac{252}{36} = 7$$

Example 10.12 : The probability of success of an event is p and that of failure is q. Find the expected number of trials to get a first success.

Solution: Let X be the random variable denoting 'Number of trials to get a first success'. The success can occur in the 1st trial. $\therefore$ The probability of success in the 1st trial is p. The success in the 2nd trial means failure in the 1st trial. $\therefore$ Probability is qp.

Success in the 3rd trial means failure in the first two trials. $\therefore$ Probability of success in the 3rd trial is q^2p. As it goes on, the success may occur in the nth trial which mean the first (n –1) trials are failures. $\therefore$ probability = $q^{n-1}p$.

$\therefore$ The probability distribution is as follows

X	1	2	3	...	n ...
$P(x)$	p	qp	q^2p	...	$q^{n-1}\ p$...

$$\therefore\ E(X) = \sum p_i\ x_i$$
$$= 1\,.\,p + 2qp + 3q^2p + \ldots + nq^{n-1}\,p\ldots$$
$$= p[1 + 2q + 3q^2 + \ldots + nq^{n-1} + \ldots]$$
$$= p[1-q]^{-2} = p(p)^{-2} = \frac{p}{p^2} = \frac{1}{p}$$

Example 10.13 : An urn contains 4 white and 3 Red balls. Find the probability distribution of the number of red balls in three draws when a ball is drawn at random with replacement. Also find its mean and variance.

Solution : The required probability distribution is [Refer Example 10.3]

X	0	1	2	3
$P(X = x)$	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$

$$\text{Mean } E(X) = \sum p_i\ x_i$$
$$= 0\left(\frac{64}{343}\right) + 1\left(\frac{144}{343}\right) + 2\left(\frac{108}{343}\right) + 3\left(\frac{27}{343}\right) = \frac{9}{7}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$
$$E(X^2) = \sum p_i\ x_i^2$$
$$= 0\left(\frac{64}{343}\right) + 1^2\left(\frac{144}{343}\right) + 2^2\left(\frac{108}{343}\right) + 3^2\left(\frac{27}{343}\right) = \frac{117}{49}$$
$$\text{Variance} = \frac{117}{49} - \left(\frac{9}{7}\right)^2 = \frac{36}{49}$$

Example 10.14 : A game is played with a single fair die, A player wins Rs. 20 if a 2 turns up, Rs. 40 if a 4 turns up, loses Rs. 30 if a 6 turns up. While he neither wins nor loses if any other face turns up. Find the expected sum of money he can win.

Solution : Let X be the random variable denoting the amount he can win. The possible values of X are 20,40, – 30 and 0.

$$P[X = 20] = P(\text{getting } 2) = \frac{1}{6}$$
$$P[X = 40] = P(\text{getting } 4) = \frac{1}{6}$$
$$P[X = -30] = P(\text{getting } 6) = \frac{1}{6}$$

The remaining probability is $\frac{1}{2}$

X	20	40	–30	0
$P(x)$	1/6	1/6	1/6	1/2

Mean $E\,(\text{X}) = \sum p_i\ x_i$

$$= 20\left(\frac{1}{6}\right) + 40\left(\frac{1}{6}\right) + (-30)\left(\frac{1}{6}\right) + 0\left(\frac{1}{2}\right) = 5$$

Expected sum of money he can win = Rs. 5

Expectation of a continuous Random Variable :

Definition : Let X be a continuous random variable with probability density function $f(x)$. Then the mathematical expectation of X is defined as

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

Note : If φ is function such that $\varphi(X)$ is a random variable and E $[\varphi\,(X)]$ exists then

$$E[\varphi\,(X)] = \int_{-\infty}^{\infty} \varphi\,(x)\,f(x)\,dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)\,dx$$

$$\text{Variance of } X = E(X^2) - [E(X)]^2$$

Results : (1) E(c) = c where c is a constant

$$\text{E}(c) = \int_{-\infty}^{\infty} c\,f(x)\,dx = c\int_{-\infty}^{\infty} f(x)\,dx = c \qquad \text{as} \int_{-\infty}^{\infty} f(x)\,dx = 1$$

(2) $E(aX \pm b) = a\,E(X) \pm b$

$$E(aX \pm b) = \int_{-\infty}^{\infty} (ax \pm b)\,f(x)\,dx = \int_{-\infty}^{\infty} ax\,f(x)\,dx \pm \int_{-\infty}^{\infty} b\,f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\,f(x)\,dx \pm b\int_{-\infty}^{\infty} f(x)\,dx = a\,E(X) \pm b$$

Example 10.15 : In a continuous distribution the p.d.f of X is

$$f(x) = \begin{cases} \frac{3}{4}x(2-x), & 0 < x < 2 \\ 0 & \text{otherwise}. \end{cases}$$

Find the mean and the variance of the distribution.

Solution :

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^2 x.\frac{3}{4}x(2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 x^2(2-x)\, dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\, dx$$

$$= \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{2}{3}(8) - \frac{16}{4}\right] = 1$$

$\therefore$ Mean = 1

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^2 x^2\frac{3}{4}x(2-x)\, dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\, dx = \frac{3}{4}\left[2\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left[\frac{16}{2} - \frac{32}{5}\right] = \frac{6}{5}$$

$$\text{Variance} = E(X^2) - [E(X)]^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

Example 10.16 : Find the mean and variance of the distribution

$$f(x) = \begin{cases} 3e^{-3x}, & 0 < x < \infty \\ 0 & \text{,elsewhere} \end{cases}$$

Solution :

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$\int_0^{\infty} x^n e^{-\alpha x}\, dx = \frac{\lfloor n}{\alpha^{n+1}}$$

When n is a positive integer

$$= \int_0^{\infty} x(3e^{-3x})\, dx = 3\int_0^{\infty} x\, e^{-3x}\, dx = 3.\frac{\lfloor 1}{3^2} = \frac{1}{3}$$

$$E(X^2) = \int_0^{\infty} x^2(3e^{-3x})\, dx = 3\int_0^{\infty} x^2 e^{-3x}\, dx = 3\,.\frac{\lfloor 2}{3^3} = \frac{2}{9}$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\therefore \text{Mean} = \frac{1}{3} \quad ; \quad \text{Variance} = \frac{1}{9}$$

EXERCISE 10.2

(1) A die is tossed twice. A success is getting an odd number on a toss. Find the mean and the variance of the probability distribution of the number of successes.

(2) Find the expected value of the number on a die when thrown.

(3) In an entrance examination a student has to answer all the 120 questions. Each question has four options and only one option is correct. A student gets 1 mark for a correct answer and loses half mark for a wrong answer. What is the expectation of the mark scored by a student if he chooses the answer to each question at random?

(4) Two cards are drawn with replacement from a well shuffled deck of 52 cards. Find the mean and variance for the number of aces.

(5) In a gambling game a man wins Rs.10 if he gets all heads or all tails and loses Rs.5 if he gets 1 or 2 heads when 3 coins are tossed once. Find his expectation of gain.

(6) The probability distribution of a random variable X is given below :

X	0	1	2	3
$P(X = x)$	0.1	0.3	0.5	0.1

If $Y = X^2 + 2X$ find the mean and variance of Y.

(7) Find the Mean and Variance for the following probability density functions

(i) $f(x) = \begin{cases} \frac{1}{24} & ,-12 \le x \le 12 \\ 0 & ,\text{otherwise} \end{cases}$ (ii) $f(x) = \begin{cases} \alpha e^{-\alpha x} & , \text{if } x > 0 \\ 0 & ,\text{otherwise} \end{cases}$

(iii) $f(x) = \begin{cases} xe^{-x} & , \text{if } x > 0 \\ 0 & ,\text{otherwise} \end{cases}$

10.4 Theoretical Distributions :

The values of random variables may be distributed according to some definite probability law which can be expressed mathematically and the corresponding probability distribution is called theoretical distribution. Theoretical distributions are based on expectations on the basis of previous experience.

In this section we shall study **(1) Binomial distribution (2) Poisson distribution (3) Normal distribution** which figure most prominently in statistical theory and in application. The first two distributions are discrete probability distributions and the third is a continuous probability distribution.

Discrete Distributions :

Binomial Distribution :

This was discovered by a Swiss Mathematician James Bernoulli (1654–1705)

Bernoulli's Trials :

Consider a random experiment that has only two possible outcomes. For example when a coin is tossed we can take the falling of head as success and falling of tail as failure. Assume that these outcomes have probabilities p and q respectively such that $p + q = 1$. If the experiment is repeated 'n' times independently with two possible outcomes they are called Bernoulli's trials. A Binomial distribution can be used under the following condition.

(i) any trial, result in a success or a failure

(ii) There are a finite number of trials which are independent.

(iii) The probability of success is the same in each trial.

Probability function of Binomial Distribution :

Let n be a given positive integer and p be a given real number such that $0 \leq p \leq 1$. Also let $q = 1 - p$. Consider the finite probability distribution described by the following table.

x_i	0	1	2	. . .	n
$P(x_i)$	q^n	$nc_1 pq^{n-1}$	$nc_2\, p^2 q^{n-2}$	. . .	p^n

The table shown above is called the Binomial distribution. The 2nd row of the table are the successive terms in the binomial expansion of $(q + p)^n$.

Binomial probability function $B(n,p,x)$ gives the probability of exactly x successes in 'n' Bernoullian trials, p being the probability of success in a trial. The constants n and p are called the parameters of the distribution.

Definition of Binomial Distribution :

A random variable X is said to follow Binomial distribution if its probability mass function is given by

$$P(X = x) = p(x) = \begin{cases} {}^{n}C_{x}\, p^{x}\, q^{n-x}, & x = 0, 1, \ldots n \\ 0 & \text{otherwise} \end{cases}$$

Constants of Binomial Distribution :

$$\text{Mean} = np$$

$$\text{Variance} = npq$$

$$\text{Standard deviation} = \sqrt{\text{variance}} = \sqrt{npq}$$

$X \sim B(n, p)$ denotes that the random variable X follows Binomial distribution with parameters n and p.

Note : In a Binomial distribution mean is always greater than the variance.

Example 10.17 : Let X be a binomially distributed variable with mean 2 and standard deviation $\frac{2}{\sqrt{3}}$. Find the corresponding probability function.

Solution : $np = 2 \quad ; \quad \sqrt{npq} = \frac{2}{\sqrt{3}}$

$$\therefore npq = 4/3$$

$$\therefore q = \frac{npq}{np} = \frac{4/3}{2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

$$np = 2 \quad \therefore \quad n\left(\frac{1}{3}\right) = 2 \Rightarrow n = 6$$

$\therefore$ The probability function for the distribution is

$$P[X = x] = 6C_{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{6-x}, x = 0, 1, 2, \quad \ldots 6$$

Example 10.18 : A pair of dice is thrown 10 times. If getting a doublet is considered a success find the probability of (i) 4 success (ii) No success.

Solution : $n = 10$. A doublet can be obtained when a pair of dice thrown is {(1,1), (2,2) (3,3), (4,4), (5,5) (6,6)} *ie.*, 6 ways.

Probability of success is getting a doublet

$$\therefore p = \frac{6}{36} = \frac{1}{6} \quad ; \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

Let X be the number of success.

$$\text{We have } P[X = x] = {}^nC_x\ p^x\ q^{n-x}$$

(a) $$P(4\text{ successes}) = P[X = 4] = {}^{10}C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6$$

$$= \frac{210 \times 5^6}{6^{10}} = \frac{35}{216}\left(\frac{5}{6}\right)^6$$

(b) $$P\text{ (no success)} = P(X = 0)$$

$$= {}^{10}C_0 \left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10}$$

Example 10.19 : In a Binomial distribution if $n = 5$and $P(X = 3) = 2P(X = 2)$ find p

Solution : $$P(X = x) = {}^nC_x\ p^x\ q^{n-x}$$

$$P(X = 3) = {}^5C_3\, p^3q^2 \quad \text{and} \quad P(X = 2) = {}^5C_2\, p^2q^3$$

$$\therefore\ {}^5C_3\ p^3q^2 = 2\left({}^5C_2\ p^2q^3\right)$$

$$\therefore\ p = 2q$$

$$p = 2(1-p) \quad \Rightarrow\ 3p = 2 \ ;\ p = \frac{2}{3}$$

Example 10.20 : If the sum of mean and variance of a Binomial Distribution is 4.8 for 5 trials find the distribution.

Solution : $$np + npq = 4.8 \ \Rightarrow\ np(1+q) = 4.8$$

$$5p\,[1 + (1-p) = 4.8$$

$$p^2 - 2p + 0.96 = 0 \quad \Rightarrow\ p = 1.2\ ,\ 0.8$$

$\therefore\ p = 0.8\ ;\ q = 0.2$ [$\because p$ cannot be greater than 1]

$\therefore$ The Binomial distribution is $P[X = x] = {}^5C_x\ (0.8)^x\ (0.2)^{5-x}$, $x = 0$ to 5

Example 10.21 : The difference between the mean and the variance of a Binomial distribution is 1 and the difference between their squares is 11.Find n.

Solution : Let the mean be $(m + 1)$ and the variance be m from the given data.[Since mean > variance in a binomial distribution]

$$(m+1)^2 - m^2 = 11 \Rightarrow\ m = 5$$

$$\therefore\ \text{mean} = m + 1 = 6$$

$$\Rightarrow\ np = 6\ ;\ npq = 5 \quad \therefore\ q = \frac{5}{6},\ p = \frac{1}{6} \quad \Rightarrow\ n = 36.$$

EXERCISE 10.3

(1) The mean of a binomial distribution is 6 and its standard deviation is 3. Is this statement true or false? Comment.

(2) A die is thrown 120 times and getting 1 or 5 is considered a success. Find the mean and variance of the number of successes.

(3) If on an average 1 ship out of 10 do not arrive safely to ports. Find the mean and the standard deviation of ships returning safely out of a total of 500 ships

(4) Four coins are tossed simultaneously. What is the probability of getting *(a)* exactly 2 heads *(b)* at least two heads *(c)* at most two heads.

(5) The overall percentage of passes in a certain examination is 80. If 6 candidates appear in the examination what is the probability that atleast 5 pass the examination.

(6) In a hurdle race a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down less than 2 hurdles.

10.4.2 Poisson Distribution :

It is named after the French Mathematician *Simeon Denis Poisson* (1781 – 1840) who discovered it. Poisson distribution is also a discrete distribution.

Poisson distribution is a limiting case of Binomial distribution under the following conditions.

(i) n the number of trials is indefinitely large ie., $n \to \infty$.

(ii) p the constant probability of success in each trial is very small ie., $p \to 0$.

(iii) $np = \lambda$ is finite where λ is a positive real number. When an event occurs rarely, the distribution of such an event may be assumed to follow a Poisson distribution.

Definition : A random variable X is said to have a Poisson distribution if the probability mass function of X is $P(X = x) = \frac{e^{-\lambda}\lambda^x}{\underline{|x}}$, $x = 0,1,2, \ldots$for some $\lambda > 0$

The mean of the Poisson Distribution is λ, and the variance is also λ.

The parameter of the Poisson distribution is λ.

Examples of Poisson Distribution :

(1) The number of alpha particles emitted by a radio active source in a given time interval.

(2) The number of telephone calls received at a telephone exchange in a given time interval.

(3) The number of defective articles in a packet of 100, produced by a good industry.

(4) The number of printing errors at each page of a book by a good publication.

(5) The number of road accidents reported in a city at a particular junction at a particular time.

Example 10.22 : Prove that the total probability is one.

Solution :

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{\lfloor x} = \frac{e^{-\lambda}\lambda^{0}}{\lfloor 0} + \frac{e^{-\lambda}\lambda^{1}}{\lfloor 1} + \frac{e^{-\lambda}\lambda^{2}}{\lfloor 2} + \ldots$$

$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{\lfloor 2} + \ldots\right] = e^{-\lambda} \cdot e^{\lambda} = e^{0} = 1$$

Example 10.23 : If a publisher of non-technical books takes a great pain to ensure that his books are free of typological errors, so that the probability of any given page containing atleast one such error is 0.005 and errors are independent from page to page (i) what is the probability that one of its 400 page novels will contain exactly one page with error. (ii) atmost three pages with errors. [e^{-2} = 0.1353 ; $e^{-0.2}$. = 0.819].

Solution :

$$n = 400 \ , \ p = 0.005$$

$$\therefore \ np = 2 = \lambda$$

(i) P(one page with error) $= P(X = 1)$

$$= \frac{e^{-\lambda}\lambda^{1}}{\lfloor 1} = \frac{e^{-2}2^{1}}{\lfloor 1} = 0.1363 \times 2 = 0.2726$$

(ii) P(atmost 3 pages with error) $= P(X \le 3)$

$$= \sum_{x=0}^{3} \frac{e^{-\lambda}\lambda^{x}}{\lfloor x} = \sum_{0}^{3} \frac{e^{-2}(2)^{x}}{\lfloor x} = e^{2}\left[1 + \frac{2}{\lfloor 1} + \frac{2^2}{\lfloor 2} + \frac{2^3}{\lfloor 3}\right]$$

$$= e^{-2}\left(\frac{19}{3}\right) = 0.8569$$

Example 10.24 : Suppose that the probability of suffering a side effect from a certain vaccine is 0.005. If 1000 persons are inoculated, find approximately the probability that (i) atmost 1 person suffer. (ii) 4, 5 or 6 persons suffer.

$[e^{-5} = 0.0067]$

Solution : Let the probability of suffering from side effect be p

$n = 1000$, $p = 0.005$, $\lambda = np = 5$.

(i) P(atmost 1 person suffer) $= p(X \leq 1)$

$$= p(X = 0) + p(X = 1)$$

$$= \frac{e^{-\lambda}\lambda^0}{\lfloor 0} + \frac{e^{-\lambda}\lambda^1}{\lfloor 1} = e^{-\lambda}[1 + \lambda]$$

$$= e^{-5}(1 + 5) = 6 \times e^{-5}$$

$$= 6 \times 0.0067 = 0.0402$$

(ii) P(4, 5 or 6 persons suffer) $= p(X = 4) + p(X = 5) + p(X = 6)$

$$= \frac{e^{-\lambda}\lambda^4}{\lfloor 4} + \frac{e^{-\lambda}\lambda^5}{\lfloor 5} + \frac{e^{-\lambda}\lambda^6}{\lfloor 6} = \frac{e^{-\lambda}\lambda^4}{\lfloor 4}\left[1 + \frac{\lambda}{5} + \frac{\lambda^2}{30}\right]$$

$$= \frac{e^{-5}5^4}{24}\left[1 + \frac{5}{5} + \frac{25}{30}\right] = \frac{e^{-5}5^4}{24}\left[\frac{17}{6}\right] = \frac{10625}{144} \times 0.0067$$

$$= 0.4944$$

Example 10.25 : In a Poisson distribution if $P(X = 2) = P(X = 3)$ find $P(X = 5)$

[given $e^{-3} = 0.050$].

Solution : Given $P(X = 2) = P(X = 3)$

$$\therefore \frac{e^{-\lambda}\lambda^2}{\lfloor 2} = \frac{e^{-\lambda}\lambda^3}{\lfloor 3}$$

$$\Rightarrow 3\lambda^2 = \lambda^3$$

$\Rightarrow \lambda^2(3 - \lambda) = 0$ As $\lambda \neq 0$. $\lambda = 3$

$$P(X = 5) = \frac{e^{-\lambda}\lambda^5}{\lfloor 5} = \frac{e^{-3}(3)^5}{\lfloor 5} = \frac{0.050 \times 243}{120} = 0.101$$

Example 10.26 : If the number of incoming buses per minute at a bus terminus is a random variable having a Poisson distribution with λ=0.9, find the probability that there will be

(i) Exactly 9 incoming buses during a period of 5 minutes

(ii) Fewer than 10 incoming buses during a period of 8 minutes.

(iii) Atleast 14 incoming buses during a period of 11 minutes.

Solution :

(i) $\left.\begin{array}{r}\lambda \text{ for number of incoming} \\ \text{buses per minute}\end{array}\right\} = 0.9$

$\left.\begin{array}{r}\therefore \lambda \text{ for number of incoming} \\ \text{buses per 5 minutes}\end{array}\right\} = 0.9 \times 5 = 4.5$

$\left.\begin{array}{r}P \text{ exactly 9 incoming buses} \\ \text{during 5 minutes}\end{array}\right\} = \dfrac{e^{-\lambda}\,\lambda^9}{\lfloor 9}$

$$\text{i.e., } P(X = 9) = \frac{e^{-4.5} \times (4.5)^9}{\lfloor 9}$$

(ii) $\left.\begin{array}{r}\text{fewer than 10 incoming buses} \\ \text{during a period of 8 minutes}\end{array}\right\} = P(X < 10)$

$$\text{Here } \lambda = 0.9 \times 8 = 7.2$$

$$\therefore \text{ Required probability } = \sum_{x=0}^{9} \frac{e^{-7.2} \times (7.2)^x}{\lfloor x}$$

(iii) $\left.\begin{array}{r}\text{P atleast 14 incoming buses} \\ \text{during a period of 11 minutes}\end{array}\right\} = P(X \geq 14) = 1 - P(X < 14)$

$$\text{Here } \lambda = 11 \times 0.9 = 9.9$$

$$\therefore \text{ Required probability } = 1 - \sum_{x=0}^{13} \frac{e^{-9.9} \times (9.9)^x}{\lfloor x}$$

(The answer can be left at this stage).

EXERCISE 10.4

(1) Let X have a Poisson distribution with mean 4. Find (i) $P(X \leq 3)$ (ii) $P(2 \leq X < 5)$ $[e^{-4} = 0.0183]$.

(2) If the probability of a defective fuse from a manufacturing unit is 2% in a box of 200 fuses find the probability that

(i) exactly 4 fuses are defective (ii) more than 3 fuses are defective $[e^{-4} = 0.0183]$.

(3) 20% of the bolts produced in a factory are found to be defective. Find the probability that in a sample of 10 bolts chosen at random exactly 2 will be defective using (i) Binomial distribution (ii) Poisson distribution. [$e^{-2} = 0.1353$].

(4) Alpha particles are emitted by a radio active source at an average rate of 5 in a 20 minutes interval. Using Poisson distribution find the probability that there will be (i) 2 emission (ii) at least 2 emission in a particular 20 minutes interval. [$e^{-5} = 0.0067$].

(5) The number of accidents in a year involving taxi drivers in a city follows a Poisson distribution with mean equal to 3. Out of 1000 taxi drivers find approximately the number of drivers with (i) no accident in a year (ii) more than 3 accidents in a year [$e^{-3} = 0.0498$].

10.4.3 Normal Distribution :

The Binomial and the Poisson distribution described above are the most useful theoretical distribution for discrete variables i.e., they relate to the occurrence of distinct events. In order to have mathematical distribution suitable for dealing with quantities whose magnitude is continuously varying, a continuous distribution is needed. The normal distribution is also called the normal probability distribution, happens to be the most useful theoretical distribution for continuous variables. Many statistical data concerning business and economic problems are displayed in the form of normal distribution. In fact normal distribution is the 'corner stone' of Modern statistics.

Like the Poisson distribution, the normal distribution may also be regarded as a limiting case of binomial distribution. Indeed when n is large and neither p nor q is close to zero the Binomial distribution is approximated by the normal distribution inspite of the fact that the former is a discrete distribution, where as the later is a continuous distribution. Examples include measurement errors in scientific experiments, anthropometric measurements of fossils, reaction times in psychological experiment, measurements of intelligence and aptitude, scores on various tests and numerous economic measures and indication.

Definition : A continuous random variable X is said to follow a normal distribution with parameter μ and σ (or μ and σ^2) if the probability function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ; \; -\infty < x < \infty, \; -\infty < \mu < \infty, \text{ and } \sigma > 0.$$

$X \sim N(\mu, \sigma)$ denotes that the random variable X follows normal distribution with mean μ and standard deviation σ.

Note : Even we can write the normal distribution as $X \sim N(\mu, \sigma^2)$ symbolically. In this case the parameters are mean and variance.

The normal distribution is also called Gaussian Distribution. The normal distribution was first discovered by De-Moivre (1667 − 1754) in 1733 as a limiting case of Binomial distribution. It was also known to Laplace not later than 1744 but through a historical error it has been credited to Gauss who first made reference to it in 1809.

Constants of Normal distribution :

Mean = μ

Variance = σ^2

Standard deviation = σ

The graph of the normal curve is shown above.

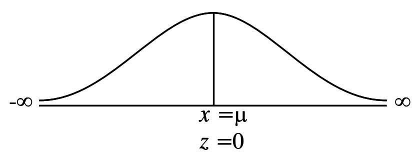


Fig. 10.3

Properties of Normal Distribution :

(1) The normal curve is bell shaped

(2) It is symmetrical about the line $X = \mu$ *ie.,* about the mean line.

(3) Mean = Median = Mode = μ

(4) The height of the normal curve is maximum at $X = \mu$ and $\frac{1}{\sigma\sqrt{2\pi}}$ is the maximum height (probability).

(5) It has only one mode at $X = \mu$. $\therefore$ The normal curve is unimodal

(6) The normal curve is asymptotic to the base line.

(7) The points of inflection are at $X = \mu \pm \sigma$

(8) Since the curve is symmetrical about $X = \mu$, the skewness is zero.

(9) Area property :

$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$

$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$

(10) A normal distribution is a close approximation to the binomial distribution when n, the number of trials is very large and p the probability of success is close to 1/2 i.e., neither p nor q is so small.

(11) It is also a limiting form of Poisson distribution i.e., as $\lambda \to \infty$ Poisson distribution tends to normal distribution.

Standard Normal Distribution :

A random variable X is called a standard normal variate if its mean is zero and its standard deviation is unity.

A normal distribution with mean μ and standard deviation σ can be converted into a standard normal distribution by performing change of scale and origin.

The formula that enables us to change from the x scale to the z – scale and vice versa is $Z = \frac{X-\mu}{\sigma}$

The probability density function of the standard normal variate Z is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad ; \; -\infty < z < \infty$$

The distribution does not contain any parameter. The standard normal distribution is denoted by $N(0,1)$.

The total area under the normal probability curve is unity.

i.e., $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \varphi(z)dz = 1 \Rightarrow \int_{-\infty}^{0} \varphi(z)dz = \int_{0}^{\infty} \varphi(z)dz = 0.5$

Area Property of Normal Distribution :

The Probability that a random variable X lies in the interval $(\mu - \sigma, \mu + \sigma)$ is given by

$$P(\mu - \sigma < X < \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\,dx$$

substituting $X = \mu - \sigma$ and $X = \mu + \sigma$ in $Z = \frac{X-\mu}{\sigma}$

$$P(-1 < Z < 1) = \int_{-1}^{1} \varphi(z)dz$$

$= 2\int_{0}^{1} \varphi(z)dz$ (by symmetry)

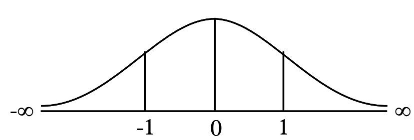


Fig. 10.4

$= 2 \times 0.3413$, (from the area table)

$= 0.6826$

Also $P(\mu - 2\sigma < X < \mu + 2\sigma)$

$$= \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)\, dx$$

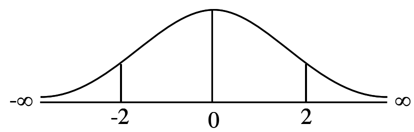


Fig. 10.5

$$P(-2 < Z < 2) = \int_{-2}^{2} \varphi(z)dz$$

$$= 2\int_{0}^{2} \varphi(z)dz \text{ , (by symmetry)}$$

$$= 2 \times 0.4772 = 0.9544$$

Similarly $P(\mu - 3\sigma < X < \mu + 3\sigma)$

$$= \int_{\mu-3\sigma}^{\mu+3\sigma} f(x)\, dx = \int_{-3}^{3} \varphi(z)dz$$

$$= 2 \times 0.49865 = 0.9973$$

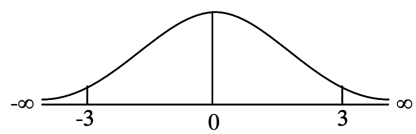


Fig. 10.6

Therefore the probability that a normal variate X lies outside the range $\mu \pm 3\sigma$ is given by

$$P(\,|X - \mu| > 3\sigma) = P(\,|Z| > 3) = 1 - p(-3 < Z < 3) = 1 - 0.9973 = 0.0027$$

Note : Since the areas under the normal probability curve have been tabulated interms of the standard normal variate Z, for any problem first convert X to Z. The entries in the table gives the areas under the normal curve between the mean ($z = 0$) and the given value of z as shown below :

Therefore entries corresponding to negative values are unnecessary because the normal curve is symmetrical. For example

$$P(0 \le Z \le 1.2) = P(-1.2 \le Z \le 0)$$

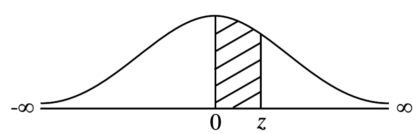


Fig. 10.7

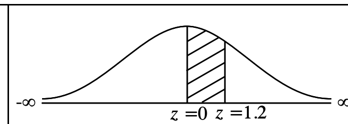
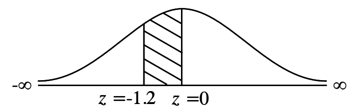
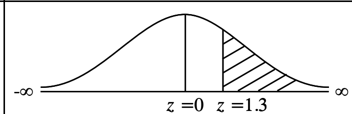
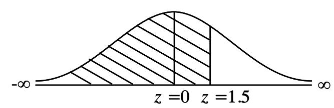
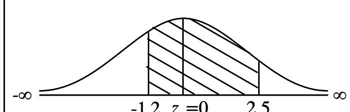
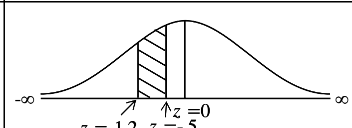
Example 10.27 : Let Z be a standard normal variate. Calculate the following probabilities.

(i) $P(0 \le Z \le 1.2)$ (ii) $P(-1.2 \le Z \le 0)$

(iii) Area to the right of $Z = 1.3$ (iv) Area to the left of $Z = 1.5$

(v) $P(-1.2 \le Z \le 2.5)$ (vi) $P(-1.2 \le Z \le -0.5)$ (vii) $P(1.5 \le Z \le 2.5)$

Solution :

(i) $P(0 \le Z \le 1.2)$ $P(0 \le Z \le 1.2)$ = area between $Z = 0$ and $Z = 1.2$ = 0.3849	 *Fig. 10.8*
(ii) $P(-1.2 \le Z \le 0)$ $P(-1.2 \le Z \le 0)$ = $P(0 \le Z \le 1.2)$ by symmetry = 0.3849	 *Fig. 10.9*
(iii) Area to the right of $Z = 1.3$ $P(Z > 1.3)$ = area between $Z = 0$ to $Z = \infty$ – area between Z = 0 to Z = 1.3 = $P(0 < Z < \infty) - P(0 \le Z < 1.3)$ = $0.5 - 0.4032 = 0.0968$	 *Fig. 10.10*
(iv) Area of the left of $Z = 1.5$ = $P(Z < 1.5)$ = $P(-\infty < Z < 0) + P(0 \le Z < 1.5)$ = $0.5 + 0.4332$ = 0.9332	 *Fig. 10.11*
(v) $P(-1.2 \le Z < 2.5)$ = $P(-1.2 < Z < 0) + P(0 < Z < 2.5)$ = $P(0 \le Z < 1.2) + P(0 \le Z \le 2.5)$ [by symmetry] = $0.3849 + 0.4938$ = 0.8787	 *Fig. 10.12*
(vi) $P(-1.2 \le Z \le -0.5)$ = $P(-1.2 < Z < 0) - P(-0.5 < Z < 0)$ = $P(0 < Z < 1.2) - P(0 < Z < 0.5)$ [due to symmetry] = $0.3849 - 0.1915 = 0.1934$	 *Fig. 10.13*

(vii) $P(1.5 \leq Z \leq 2.5)$

Required area

$= P(0 \leq Z \leq 2.5) - P(0 \leq Z \leq 1.5) = 0.4938 - 0.4332 = 0.0606$

Example 10.28 : Let Z be a standard normal variate. Find the value of c in the following problems.

(i) $P(Z < c) = 0.05$ (ii) $P(-c < Z < c) = 0.94$

(iii) $P(Z > c) = 0.05$ (iv) $P(c < Z < 0) = 0.31$

Solution :

(i) $P(Z < c) = 0.05$ i.e., $P(-\infty < Z < c) = 0.05$

As area is < 0.5, c lies to the left of $Z = 0$.

From the area table Z value for the area 0.45 is 1.65. $\therefore c = -1.65$

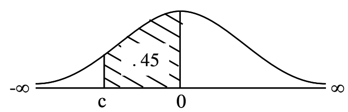


Fig. 10.14

(ii) $P(-c < Z < c) = 0.94$

As $Z = -c$ and $Z = +c$ lie at equal distance from $Z = 0$,

$\therefore$ We have $P(0 < Z < c) = \frac{0.94}{2} = 0.47$.

Z value for the area 0.47 from the table is 1.88 $\therefore$ $c = 1.88$ and $-c = -1.88$

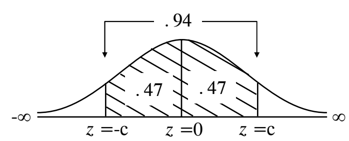


Fig. 10.15

(iii) $P(Z > c) = 0.05 \Rightarrow P(c < Z < \infty) = 0.05$

From the data it is clear that c lies to the right of $Z = 0$

The area to the right of $Z = 0$ is 0.5

$$P(0 < Z < \infty) - P(0 < Z < c) = 0.05$$
$$0.5 - P(0 < Z < c) = 0.05$$
$$\therefore 0.5 - 0.05 = P(0 < Z < c)$$
$$0.45 = P(0 < Z < c)$$

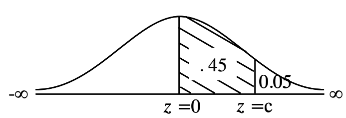


Fig. 10.16

From the area table Z value for the area 0.45 is 1.65 $\therefore$ $c = 1.65$

(iv) $P(c < Z < 0) = 0.31$

As c is less than zero, it lies to the left of $Z = 0$. From the area table the Z value for the area 0.31 is 0.88. As it in to the left of $Z = 0$, $c = -0.88$

Example 10.29 : If X is normally distributed with mean 6 and standard deviation 5 find. (i) $P(0 \leq X \leq 8)$ (ii) $P(|X - 6| < 10)$

Solution : Given $\mu = 6$, $\sigma = 5$

(i) $P(0 \le X \le 8)$

We know that $Z = \dfrac{X-\mu}{\sigma}$

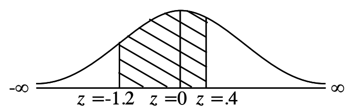


Fig. 10.17

When $X = 0$, $Z = \dfrac{0-6}{5} = \dfrac{-6}{5} = -1.2$

When $X = 8$, $Z = \dfrac{8-6}{5} = \dfrac{2}{5} = 0.4$

$\therefore P(0 \le X \le 8) = P(-1.2 < Z < 0.4)$

$= P(0 < Z < 1.2) + P(0 < Z < .4)$ (due to symmetry)

$= 0.3849 + 0.1554$

$= 0.5403$

(ii) $P(|X-6| < 10) = P(-10 < (X-6) < 10) \Rightarrow P(-4 < X < 16)$

When $X = -4$, $Z = \dfrac{-4-6}{5} = \dfrac{-10}{5} = -2$

When $X = 16$, $Z = \dfrac{16-6}{5} = \dfrac{10}{5} = 2$

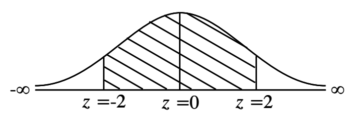


Fig. 10.18

$P(-4 < X < 16) = P(-2 < Z < 2)$

$= 2P(0 < Z < 2)$ (due to symmetry)

$= 2(0.4772) = 0.9544$

Example 10.30 : The mean score of 1000 students for an examination is 34 and S.D is 16. (i) How many candidates can be expected to obtain marks between 30 and 60 assuming the normality of the distribution and (ii) determine the limit of the marks of the central 70% of the candidates.

Solution : $\mu = 34$, $\sigma = 16$, $N = 1000$

(i) $P(30 < X < 60)$; $Z = \dfrac{X-\mu}{\sigma}$

$\therefore X = 30$, $Z_1 = \dfrac{30-\mu}{\sigma} = \dfrac{30-34}{16}$

$= \dfrac{-4}{16} = -0.25$

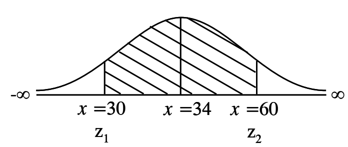


Fig. 10.19

$Z_1 = -0.25$

$Z_2 = \dfrac{60-34}{16} = \dfrac{26}{16} = 1.625$

$Z_2 \approx 1.63$ (app.)

$P(-0.25 < Z < 1.63) = P(0 < Z < 0.25) + P(0 < Z < 1.63)$ (due to symmetry)

$= 0.0987 + 0.4484 = 0.5471$

No of students scoring between 30 and 60 $\Big\} = 0.5471 \times 1000 = 547.$

(ii) limit of central 70% of Candidates **:**

Value of Z_1 from the area table for the area 0.35 $\Big\} = -1.04$

[as Z_1 lies to left of $Z = 0$]

Similarly $Z_2 = 1.04$

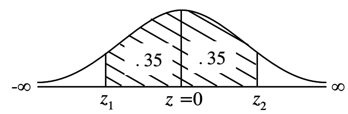


Fig. 10.20

$$Z_1 = \frac{X - 34}{16} = 1.04 \qquad\qquad Z_2 = \frac{X - 34}{16} = -1.04$$

$$X_1 = 16 \times 1.04 + 34 \qquad\qquad X_2 - 34 = -1.04 \times 16 + 34$$

$$= 16.64 + 34 \qquad\qquad X_2 = -16.64 + 34$$

$$X_1 = 50.64 \qquad\qquad X_2 = 17.36$$

$\therefore$ 70% of the candidate score between 17.36 and 50.64.

Example 10.31 : Obtain k, μ and σ^2 of the normal distribution whose probability distribution function is given by

$$f(x) = k\, e^{-2x^2 + 4x} \qquad -\infty < X < \infty$$

Solution : Consider

$$-2x^2 + 4x = -2\,(x^2 - 2x) = -2\,[(x-1)^2 - 1] = -2\,(x-1)^2 + 2$$

$$\therefore\ e^{-2x^2 + 4x} = e^2 .\ e^{-2(x-1)^2}$$

$$= e^2 .\ e^{-\frac{1}{2}\,\frac{(x-1)^2}{1/4}} \qquad = e^2 .\ e^{-\frac{1}{2}\left(\frac{x-1}{1/2}\right)^2}$$

Comparing it with $f(x)$ we get

$$k\, e^{-2x^2 + 4x} = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\Rightarrow\ k\, e^2\, e^{-\frac{1}{2}\left(\frac{x-1}{1/2}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{we get } \sigma = \frac{1}{2} = \mu = 1 \text{ and } k = \frac{1}{\frac{1}{2}\sqrt{2\pi}} \,.\, e^{-2} = \frac{2e^{-2}}{\sqrt{2\pi}},\ \sigma^2 = \frac{1}{4}$$

Example 10.32 : The air pressure in a randomly selected tyre put on a certain model new car is normally distributed with mean value 31 psi and standard deviation 0.2 psi.

(i) What is the probability that the pressure for a randomly selected tyre (a) between 30.5 and 31.5 psi (b) between 30 and 32 psi

(ii) What is the probability that the pressure for a randomly selected tyre exceeds 30.5 psi ?

Solution : Given $\mu = 31$ and $\sigma = 0.2$

(i) (a) $P(30.5 < X < 31.5)$; $Z = \dfrac{X-\mu}{\sigma}$

When $X = 30.5$, $Z = \dfrac{30.5-31}{0.2} = \dfrac{-0.5}{0.2} = -2.5$

When $X = 31.5$, $Z = \dfrac{31.5-31}{0.2} = \dfrac{0.5}{0.2} = 2.5$

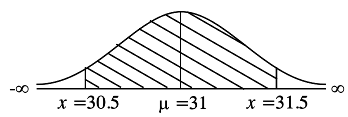


Fig. 10.21

$\therefore$ Required probability

$$\begin{aligned} P(30.5 < X < 31.5) &= P(-2.5 < Z < 2.5) \\ &= 2\,P(0 < Z < 2.5) \\ &\quad \text{[since due to symmetry]} \\ &= 2(0.4938) = 0.9876 \end{aligned}$$

(b) $P(30 < X < 32)$

When $X = 30$, $Z = \dfrac{30-31}{0.2} = \dfrac{-1}{0.2} = -5$

When $X = 32$, $Z = \dfrac{32-31}{0.2} = \dfrac{1}{0.2} = 5$

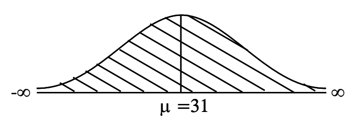


Fig. 10.22

$P(30 < X < 32) = P(-5 < Z < 5) =$ area under the whole curve $= 1$ (app.)

(ii) When $X = 30.5$, $Z = \dfrac{30.5-31}{0.2} = \dfrac{-0.5}{0.2} = -2.5$

$$\begin{aligned} P(X > 30.5) &= P(Z > -2.5) \\ &= 0.5 + p(0 < Z < 2.5) \\ &= 0.5 + 0.4938 = 0.9938 \end{aligned}$$

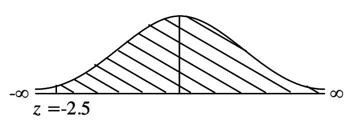


Fig. 10.23

EXERCISE 10.5

(1) If X is a normal variate with mean 80 and standard deviation 10, compute the following probabilities by standardizing.

(i) $P(X \leq 100)$ (ii) $P(X \leq 80)$

(iii) $P(65 \leq X \leq 100)$ (iv) $P(70 < X)$

(v) $P(85 \leq X \leq 95)$

(2) If Z is a standard normal variate, find the value of c for the following

(i) $P(0 < Z < c) = 0.25$ *(ii)* $P(-c < Z < c) = 0.40$

(iii) $P(Z > c) = 0.85$

(3) Suppose that the amount of cosmic radiation to which a person is exposed when flying by jet across the United States is a random variable having a normal distribution with a mean of 4.35 m rem and a standard deviation of 0.59 m rem. What is the probability that a person will be exposed to more than 5.20 m rem of cosmic radiation of such a flight.

(4) The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement within 12 months.

(5) The mean weight of 500 male students in a certain college in 151 pounds and the standard deviation is 15 pounds. Assuming the weights are normally distributed, find how many students weigh (i) between 120 and 155 pounds (ii) more than 185 pounds.

(6) If the height of 300 students are normally distributed with mean 64.5 inches and standard deviation 3.3 inches, find the height below which 99% of the student lie.

(7) Marks in an aptitude test given to 800 students of a school was found to be normally distributed. 10% of the students scored below 40 marks and 10% of the students scored above 90 marks. Find the number of students scored between 40 and 90.

(8) Find c, μ and σ^2 of the normal distribution whose probability function is given by $f(x) = c\, e^{-x^2 + 3x}$, $-\infty < X < \infty$.

OBJECTIVE TYPE QUESTIONS

Choose the correct or most suitable answer :

(1) The gradient of the curve $y = -2x^3 + 3x + 5$ at $x = 2$ is

(1) -20 (2) 27 (3) -16 (4) -21

(2) The rate of change of area A of a circle of radius r is

(1) $2\pi r$ (2) $2\pi r\dfrac{dr}{dt}$ (3) $\pi r^2\dfrac{dr}{dt}$ (4) $\pi\dfrac{dr}{dt}$

(3) The velocity v of a particle moving along a straight line when at a distance x from the origin is given by $a + bv^2 = x^2$ where a and b are constants. Then the acceleration is

(1) $\dfrac{b}{x}$ (2) $\dfrac{a}{x}$ (3) $\dfrac{x}{b}$ (4) $\dfrac{x}{a}$

(4) A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1 cm^3 / min. The rate at which the diameter is decreasing when the diameter is 10 cms is

(1) $\dfrac{-1}{50\pi}$ cm / min (2) $\dfrac{1}{50\pi}$ cm / min

(3) $\dfrac{-11}{75\pi}$ cm / min (4) $\dfrac{-2}{75\pi}$ cm / min.

(5) The slope of the tangent to the curve $y = 3x^2 + 3\sin x$ at $x = 0$ is

(1) 3 (2) 2 (3) 1 (4) -1

(6) The slope of the normal to the curve $y = 3x^2$ at the point whose x coordinate is 2 is

(1) $\dfrac{1}{13}$ (2) $\dfrac{1}{14}$ (3) $\dfrac{-1}{12}$ (4) $\dfrac{1}{12}$

(7) The point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to the x – axis is

(1) $\left(\dfrac{5}{2}, \dfrac{-17}{2}\right)$ (2) $\left(\dfrac{-5}{2}, \dfrac{-17}{2}\right)$ (3) $\left(\dfrac{-5}{2}, \dfrac{17}{2}\right)$ (4) $\left(\dfrac{3}{2}, \dfrac{-17}{2}\right)$

(8) The equation of the tangent to the curve $y = \dfrac{x^3}{5}$ at the point $(-1, {}^{-1}/_{5})$ is

(1) $5y + 3x = 2$ (2) $5y - 3x = 2$ (3) $3x - 5y = 2$ (4) $3x + 3y = 2$

(9) The equation of the normal to the curve $\theta = \frac{1}{t}$ at the point $(-3, -{}^1/_3)$ is

(1) $3\theta = 27t - 80$ (2) $5\theta = 27t - 80$

(3) $3\theta = 27t + 80$ (4) $\theta = \frac{1}{t}$

(10) The angle between the curves $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and $\frac{x^2}{8} - \frac{y^2}{8} = 1$ is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

(11) The angle between the curve $y = e^{mx}$ and $y = e^{-mx}$ for $m > 1$ is

(1) $\tan^{-1}\left(\frac{2m}{m^2-1}\right)$ (2) $\tan^{-1}\left(\frac{2m}{1-m^2}\right)$

(3) $\tan^{-1}\left(\frac{-2m}{1+m^2}\right)$ (4) $\tan^{-1}\left(\frac{2m}{m^2+1}\right)$

(12) The parametric equations of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ are

(1) $x = a\sin^3\theta$; $y = a\cos^3\theta$ (2) $x = a\cos^3\theta$; $y = a\sin^3\theta$

(3) $x = a^3\sin\theta$; $y = a^3\cos\theta$ (4) $x = a^3\cos\theta$; $y = a^3\sin\theta$

(13) If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle θ with the x – axis then the slope of the normal is

(1) $-\cot\theta$ (2) $\tan\theta$ (3) $-\tan\theta$ (4) $\cot\theta$

(14) If the length of the diagonal of a square is increasing at the rate of 0.1 cm / sec. What is the rate of increase of its area when the side is $\frac{15}{\sqrt{2}}$ cm?

(1) 1.5 cm^2/sec (2) 3 cm^2/sec (3) $3\sqrt{2}$ cm^2/sec (4) 0.15 cm^2/sec

(15) What is the surface area of a sphere when the volume is increasing at the same rate as its radius?

(1) 1 (2) $\frac{1}{2\pi}$ (3) 4π (4) $\frac{4\pi}{3}$

(16) For what values of x is the rate of increase of $x^3 - 2x^2 + 3x + 8$ is twice the rate of increase of x

(1) $\left(-\frac{1}{3}, -3\right)$ (2) $\left(\frac{1}{3}, 3\right)$ (3) $\left(-\frac{1}{3}, 3\right)$ (4) $\left(\frac{1}{3}, 1\right)$

(17) The radius of a cylinder is increasing at the rate of 2cm / sec and its altitude is decreasing at the rate of 3cm / sec. The rate of change of volume when the radius is 3cm and the altitude is 5cm is

(1) 23π (2) 33π (3) 43π (4) 53π

(18) If $y = 6x - x^3$ and x increases at the rate of 5 units per second, the rate of change of slope when $x = 3$ is

(1) -90 units / sec (2) 90 units / sec

(3) 180 units / sec (4) -180 units / sec

(19) If the volume of an expanding cube is increasing at the rate of 4cm^3 / sec then the rate of change of surface area when the volume of the cube is 8 cubic cm is

(1) 8cm^2/sec (2) 16cm^2 / sec (3) $2\ \text{cm}^2$ / sec (4) $4\ \text{cm}^2$ / sec

(20) The gradient of the tangent to the curve $y = 8 + 4x - 2x^2$ at the point where the curve cuts the y-axis is

(1) 8 (2) 4 (3) 0 (4) -4

(21) The Angle between the parabolas $y^2 = x$ and $x^2 = y$ at the origin is

(1) $2\tan^{-1}\left(\frac{3}{4}\right)$ (2) $\tan^{-1}\left(\frac{4}{3}\right)$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

(22) For the curve $x = e^t \cos t$; $y = e^t \sin t$ the tangent line is parallel to the x-axis when t is equal to

(1) $-\frac{\pi}{4}$ (2) $\frac{\pi}{4}$ (3) 0 (4) $\frac{\pi}{2}$

(23) If a normal makes an angle θ with positive x-axis then the slope of the curve at the point where the normal is drawn is

(1) $-\cot\theta$ (2) $\tan\theta$ (3) $-\tan\theta$ (4) $\cot\theta$

(24) The value of 'a' so that the curves $y = 3e^x$ and $y = \frac{a}{3}e^{-x}$ intersect orthogonally is

(1) -1 (2) 1 (3) $\frac{1}{3}$ (4) 3

(25) If $s = t^3 - 4t^2 + 7$, the velocity when the acceleration is zero is

(1) $\frac{32}{3}$ m/sec (2) $\frac{-16}{3}$ m/sec (3) $\frac{16}{3}$ m/sec (4) $\frac{-32}{3}$ m/sec

(26) If the velocity of a particle moving along a straight line is directly proportional to the square of its distance from a fixed point on the line. Then its acceleration is proportional to

(1) s (2) s^2 (3) s^3 (4) s^4

(27) The Rolle's constant for the function $y = x^2$ on $[-2, 2]$ is

(1) $\frac{2\sqrt{3}}{3}$ (2) 0 (3) 2 (4) -2

(28) The 'c' of Lagranges Mean Value Theorem for the function

$f(x) = x^2 + 2x - 1$; $a = 0$, $b = 1$ is

(1) -1 (2) 1 (3) 0 (4) $\frac{1}{2}$

(29) The value of c in Rolle's Theorem for the function $f(x) = \cos\frac{x}{2}$ on $[\pi, 3\pi]$ is

(1) 0 2) 2π (3) $\frac{\pi}{2}$ (4) $\frac{3\pi}{2}$

(30) The value of 'c' of Lagranges Mean Value Theorem for $f(x) = \sqrt{x}$ when $a = 1$ and $b = 4$ is

(1) $\frac{9}{4}$ (2) $\frac{3}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

(31) $\lim\limits_{x \to \infty} \frac{x^2}{e^x}$ is =

(1) 2 (2) 0 (3) ∞ (4) 1

(32) $\lim\limits_{x \to 0} \frac{a^x - b^x}{c^x - d^x}$

(1) ∞ (2) 0 (3) $\log\frac{ab}{cd}$ (4) $\frac{\log(a/b)}{\log(c/d)}$

(33) If $f(a) = 2$; $f'(a) = 1$; $g(a) = -1$; $g'(a) = 2$ then the value of $\lim\limits_{x \to a} \frac{g(x)\,f(a) - g(a)\,f(x)}{x - a}$ is

(1) 5 (2) -5 (3) 3 (4) -3

(34) Which of the following function is increasing in $(0, \infty)$

(1) e^x (2) $\frac{1}{x}$ (3) $-x^2$ (4) x^{-2}

(35) The function $f(x) = x^2 - 5x + 4$ is increasing in

(1) $(-\infty, 1)$ (2) $(1, 4)$ (3) $(4, \infty)$ (4) everywhere

(36) The function $f(x) = x^2$ is decreasing in

(1) $(-\infty, \infty)$ (2) $(-\infty, 0)$ (3) $(0, \infty)$ (4) $(-2, \infty)$

(37) The function $y = \tan x - x$ is

(1) an increasing function in $\left(0\,,\frac{\pi}{2}\right)$

(2) a decreasing function in $\left(0\,,\frac{\pi}{2}\right)$

(3) increasing in $\left(0\,,\frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

(4) decreasing in $\left(0\,,\frac{\pi}{4}\right)$ and increasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

(38) In a given semi circle of diameter 4 cm a rectangle is to be inscribed. The maximum area of the rectangle is

(1) 2 (2) 4 (3) 8 (4) 16

(39) The least possible perimeter of a rectangle of area $100m^2$ is

(1) 10 (2) 20 (3) 40 (4) 60

(40) If $f(x) = x^2 - 4x + 5$ on [0, 3] then the absolute maximum value is

(1) 2 (2) 3 (3) 4 (4) 5

(41) The curve $y = -e^{-x}$ is

(1) concave upward for $x > 0$ (2) concave downward for $x > 0$

(2) everywhere concave upward (4) everywhere concave downward

(42) Which of the following curves is concave down?

(1) $y = -x^2$ (2) $y = x^2$ (3) $y = e^x$ (4) $y = x^2 + 2x - 3$

(43) The point of inflexion of the curve $y = x^4$ is at

(1) $x = 0$ (2) $x = 3$ (3) $x = 12$ (4) nowhere

(44) The curve $y = ax^3 + bx^2 + cx + d$ has a point of inflexion at $x = 1$ then

(1) $a + b = 0$ (2) $a + 3b = 0$ (3) $3a + b = 0$ (4) $3a + b = 1$

(45) If $u = x^y$ then $\frac{\partial u}{\partial x}$ is equal to

(1) yx^{y-1} (2) $u \log x$ (3) $u \log y$ (4) xy^{x-1}

(46) If $u = \sin^{-1}\left(\frac{x^4 + y^4}{x^2 + y^2}\right)$ and f = sin u then f is a homogeneous function of degree

(1) 0 (2) 1 (3) 2 (4) 4

(47) If $u = \frac{1}{\sqrt{x^2 + y^2}}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to

(1) $\frac{1}{2}u$ (2) u (3) $\frac{3}{2}u$ (4) $-u$

(48) The curve $y^2 (x - 2) = x^2 (1 + x)$ has

(1) an asymptote parallel to x-axis (2) an asymptote parallel to y-axis

(3) asymptotes parallel to both axes (4) no asymptotes

(49) If $x = r \cos \theta$, $y = r \sin \theta$, then $\dfrac{\partial r}{\partial x}$ is equal to

(1) $\sec \theta$ (2) $\sin \theta$ (3) $\cos \theta$ (4) $\operatorname{cosec} \theta$

(50) Identify the true statements in the following :

(i) If a curve is symmetrical about the origin, then it is symmetrical about both axes.

(ii) If a curve is symmetrical about both the axes, then it is symmetrical about the origin.

(iii) A curve $f(x, y) = 0$ is symmetrical about the line $y = x$ if $f(x, y) = f(y, x)$.

(iv) For the curve $f(x, y) = 0$, if $f(x, y) = f(-y, -x)$, then it is symmetrical about the origin.

(1) (ii), (iii) (2) (i), (iv) (3) (i), (iii) (4) (ii), (iv)

(51) If $u = \log \left(\dfrac{x^2 + y^2}{xy}\right)$ then $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y}$ is

(1) 0 (2) u (3) $2u$ (4) u^{-1}

(52) The percentage error in the 11th root of the number 28 is approximately _____ times the percentage error in 28.

(1) $\dfrac{1}{28}$ (2) $\dfrac{1}{11}$ (3) 11 (4) 28

(53) The curve $a^2y^2 = x^2 (a^2 - x^2)$ has

(1) only one loop between $x = 0$ and $x = a$

(2) two loops between $x = 0$ and $x = a$

(3) two loops between $x = -a$ and $x = a$

(4) no loop

(54) An asymptote to the curve $y^2 (a + 2x) = x^2 (3a - x)$ is

(1) $x = 3a$ (2) $x = -a/2$ (3) $x = a/2$ (4) $x = 0$

(55) In which region the curve $y^2(a + x) = x^2 (3a - x)$ does not lie?

(1) $x > 0$ (2) $0 < x < 3a$ (3) $x \le -a$ and $x > 3a$ (4) $-a < x < 3a$

(56) If $u = y \sin x$, then $\dfrac{\partial^2 u}{\partial x \, \partial y}$ is equal to

(1) $\cos x$ (2) $\cos y$ (3) $\sin x$ 4) 0

(57) If $u = f\left(\frac{y}{x}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to

(1) 0 (2) 1 (3) $2u$ (4) u

(58) The curve $9y^2 = x^2(4 - x^2)$ is symmetrical about

(1) y-axis (2) x-axis (3) $y = x$ (4) both the axes

(59) The curve $ay^2 = x^2(3a - x)$ cuts the y-axis at

(1) $x = -3a, x = 0$ (2) $x = 0, x = 3a$ (3) $x = 0, x = a$ (4) $x = 0$

(60) The value of $\int_0^{\pi/2} \frac{\cos^{5/3}x}{\cos^{5/3}x + \sin^{5/3}x}\, dx$ is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{4}$ (3) 0 (4) π

(61) The value of $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\, dx$ is

(1) $\frac{\pi}{2}$ (2) 0 (3) $\frac{\pi}{4}$ (4) π

(62) The value of $\int_0^{1} x(1 - x)^4\, dx$ is

(1) $\frac{1}{12}$ (2) $\frac{1}{30}$ (3) $\frac{1}{24}$ (4) $\frac{1}{20}$

(63) The value of $\int_{-\pi/2}^{\pi/2} \left(\frac{\sin x}{2 + \cos x}\right) dx$ is

(1) 0 (2) 2 (3) log 2 (4) log 4

(64) The value of $\int_0^{\pi} \sin^4 x\, dx$ is

(1) $3\pi/16$ (2) $3/16$ (3) 0 (4) $3\pi/8$

(65) The value of $\int_0^{\pi/4} \cos^3 2x\, dx$ is

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) 0 (4) $\frac{2\pi}{3}$

(66) The value of $\int_0^{\pi} \sin^2 x \cos^3 x \, dx$ is

(1) π (2) $\pi/2$ (3) $\pi/4$ (4) 0

(67) The area bounded by the line $y = x$, the x-axis, the ordinates $x = 1$, $x = 2$ is

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$ (3) $\frac{1}{2}$ (4) $\frac{7}{2}$

(68) The area of the region bounded by the graph of $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{4}$ is

(1) $\sqrt{2} + 1$ (2) $\sqrt{2} - 1$ (3) $2\sqrt{2} - 2$ (4) $2\sqrt{2} + 2$

(69) The area between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its auxillary circle is

(1) $\pi b(a - b)$ (2) $2\pi a\,(a - b)$ (3) $\pi a\,(a - b)$ (4) $2\pi b\,(a - b)$

(70) The area bounded by the parabola $y^2 = x$ and its latus rectum is

(1) $\frac{4}{3}$ (2) $\frac{1}{6}$ (3) $\frac{2}{3}$ (4) $\frac{8}{3}$

(71) The volume of the solid obtained by revolving $\frac{x^2}{9} + \frac{y^2}{16} = 1$ about the minor axis is

(1) 48π (2) 64π (3) 32π (4) $128\,\pi$

(72) The volume, when the curve $y = \sqrt{3 + x^2}$ from $x = 0$ to $x = 4$ is rotated about x-axis is

(1) $100\,\pi$ (2) $\frac{100}{9}\,\pi$ (3) $\frac{100}{3}\,\pi$ (4) $\frac{100}{3}$

(73) The volume generated when the region bounded by $y = x$, $y = 1$, $x = 0$ is rotated about y-axis is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{3}$ (4) $\frac{2\pi}{3}$

(74) Volume of solid obtained by revolving the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about major and minor axes are in the ratio

(1) $b^2 : a^2$ (2) $a^2 : b^2$ (3) $a : b$ (4) $b : a$

(75) The volume generated by rotating the triangle with vertices at (0, 0), (3, 0) and (3, 3) about x-axis is

(1) 18π (2) 2π (3) 36π (4) 9π

(76) The length of the arc of the curve $x^{2/3} + y^{2/3} = 4$ is

(1) 48 (2) 24

(3) 12 (4) 96

(77) The surface area of the solid of revolution of the region bounded by $y = 2x$, $x = 0$ and $x = 2$ about x-axis is

(1) $8\sqrt{5}\,\pi$ (2) $2\sqrt{5}\,\pi$ (3) $\sqrt{5}\pi$ (4) $4\sqrt{5}\pi$

(78) The curved surface area of a sphere of radius 5, intercepted between two parallel planes of distance 2 and 4 from the centre is

(1) 20π (2) 40π (3) 10π (4) 30π

(79) The integrating factor of $\frac{dy}{dx} + 2\frac{y}{x} = e^{4x}$ is

(1) $\log x$ (2) x^2 (3) e^x (4) x

(80) If $\cos x$ is an integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$ then $P =$

(1) $-\cot x$ (2) $\cot x$ (3) $\tan x$ (4) $-\tan x$

(81) The integrating factor of $dx + xdy = e^{-y}\sec^2 y\, dy$ is

(1) e^x (2) e^{-x} (3) e^y (4) e^{-y}

(82) Integrating factor of $\frac{dy}{dx} + \frac{1}{x\log x}.y = \frac{2}{x^2}$ is

(1) e^x (2) $\log x$ (3) $\frac{1}{x}$ (4) e^{-x}

(83) Solution of $\frac{dx}{dy} + mx = 0$, where $m < 0$ is

(1) $x = ce^{my}$ (2) $x = ce^{-my}$ (3) $x = my + c$ (4) $x = c$

(84) $y = cx - c^2$ is the general solution of the differential equation

(1) $(y')^2 - xy' + y = 0$ (2) $y'' = 0$

(3) $y' = c$ (4) $(y')^2 + xy' + y = 0$

(85) The differential equation $\left(\frac{dx}{dy}\right)^2 + 5y^{1/3} = x$ is

(1) of order 2 and degree 1

(2) of order 1 and degree 2

(3) of order 1 and degree 6

(4) of order 1 and degree 3

(86) The differential equation of all non-vertical lines in a plane is

(1) $\frac{dy}{dx} = 0$ (2) $\frac{d^2y}{dx^2} = 0$ (3) $\frac{dy}{dx} = m$ (4) $\frac{d^2y}{dx^2} = m$

(87) The differential equation of all circles with centre at the origin is

(1) $x\,dy + y\,dx = 0$ (2) $x\,dy - y\,dx = 0$

(3) $x\,dx + y\,dy = 0$ (4) $x\,dx - y\,dy = 0$

(88) The integrating factor of the differential equation $\frac{dy}{dx} + py = Q$ is

(1) $\int pdx$ (2) $\int Q\,dx$ (3) $e^{\int Q\,dx}$ (4) $e^{\int pdx}$

(89) The complementary function of $(D^2 + 1)y = e^{2x}$ is

(1) $(Ax + B)e^{x}$ (2) $A \cos x + B \sin x$ (3) $(Ax + B)e^{2x}$ (4) $(Ax + B)e^{-x}$

(90) A particular integral of $(D^2 - 4D + 4)y = e^{2x}$ is

(1) $\frac{x^2}{2} e^{2x}$ (2) xe^{2x} (3) xe^{-2x} (4) $\frac{x}{2} e^{-2x}$

(91) The differential equation of the family of lines $y = mx$ is

(1) $\frac{dy}{dx} = m$ (2) $ydx - xdy = 0$

(3) $\frac{d^2y}{dx^2} = 0$ (4) $ydx + x\,dy = 0$

(92) The degree of the differential equation $\sqrt{1 + \left(\frac{dy}{dx}\right)^{1/3}} = \frac{d^2y}{dx^2}$

(1) 1 (2) 2 (3) 3 (4) 6

(93) The degree of the differential equation $c = \frac{\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{2/3}}{\frac{d^3y}{dx^3}}$ where c is a constant is

(1) 1 (2) 3 (3) -2 (4) 2

(94) The amount present in a radio active element disintegrates at a rate proportional to its amount. The differential equation corresponding to the above statement is (k is negative)

(1) $\frac{dp}{dt} = \frac{k}{p}$ (2) $\frac{dp}{dt} = kt$ (3) $\frac{dp}{dt} = kp$ (4) $\frac{dp}{dt} = -kt$

(95) The differential equation satisfied by all the straight lines in xy plane is

(1) $\frac{dy}{dx}$ = a constant (2) $\frac{d^2y}{dx^2} = 0$ (3) $y + \frac{dy}{dx} = 0$ (4) $\frac{d^2y}{dx^2} + y = 0$

(96) If $y = ke^{\lambda x}$ then its differential equation is

(1) $\frac{dy}{dx} = \lambda y$ (2) $\frac{dy}{dx} = ky$ (3) $\frac{dy}{dx} + ky = 0$ (4) $\frac{dy}{dx} = e^{\lambda x}$

(97) The differential equation obtained by eliminating a and b from $y = ae^{3x} + be^{-3x}$ is

(1) $\frac{d^2y}{dx^2} + ay = 0$ (2) $\frac{d^2y}{dx^2} - 9y = 0$ (3) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 0$ (4) $\frac{d^2y}{dx^2} + 9x = 0$

(98) The differential equation formed by eliminating A and B from the relation $y = e^x (A \cos x + B \sin x)$ is

(1) $y_2 + y_1 = 0$ (2) $y_2 - y_1 = 0$

(3) $y_2 - 2y_1 + 2y = 0$ (4) $y_2 - 2y_1 - 2y = 0$

(99) If $\frac{dy}{dx} = \frac{x-y}{x+y}$ then

(1) $2xy + y^2 + x^2 = c$ (2) $x^2 + y^2 - x + y = c$

(3) $x^2 + y^2 - 2xy = c$ (4) $x^2 - y^2 - 2xy = c$

(100) If $f'(x) = \sqrt{x}$ and $f(1) = 2$ then $f(x)$ is

(1) $-\frac{2}{3}\left(x\sqrt{x} + 2\right)$ (2) $\frac{3}{2}\left(x\sqrt{x} + 2\right)$

(3) $\frac{2}{3}\left(x\sqrt{x} + 2\right)$ (4) $\frac{2}{3}x\left(\sqrt{x} + 2\right)$

(101) On putting $y = vx$, the homogeneous differential equation $x^2dy + y(x + y)dx = 0$ becomes

(1) $xdv + (2v + v^2)dx = 0$ (2) $vdx + (2x + x^2)dv = 0$

(3) $v^2dx - (x + x^2)dv = 0$ (4) $vdv + (2x + x^2)dx = 0$

(102) The integrating factor of the differential equation $\frac{dy}{dx} - y \tan x = \cos x$ is

(1) $\sec x$ (2) $\cos x$ (3) $e^{\tan x}$ (4) $\cot x$

(103) The P.I. of $(3D^2 + D - 14)y = 13e^{2x}$ is

(1) $26x\, e^{2x}$ (2) $13x\, e^{2x}$ (3) $x\, e^{2x}$ (4) $x^2/2\, e^{2x}$

(104) The particular integral of the differential equation $f(D)y = e^{ax}$ where $f(D) = (D - a)\, g(D)$, $g(a) \neq 0$ is

(1) me^{ax} (2) $\frac{e^{ax}}{g(a)}$ (3) $g(a)e^{ax}$ (4) $\frac{xe^{ax}}{g(a)}$

(105) Which of the following are statements?

(i) May God bless you. (ii) Rose is a flower

(iii) Milk is white. (iv) 1 is a prime number

(1) (i), (ii), (iii) (2) (i), (ii), (iv) (3) (i), (iii), (iv) (4) (ii), (iii), (iv)

(106) If a compound statement is made up of three simple statements, then the number of rows in the truth table is

(1) 8 (2) 6 (3) 4 (4) 2

(107) If p is T and q is F, then which of the following have the truth value T ?

(i) $p \vee q$ (ii) $\sim p \vee q$ (iii) $p \vee \sim q$ (iv) $p \wedge \sim q$

(1) (i), (ii), (iii) (2) (i), (ii), (iv)

(3) (i), (iii), (iv) (4) (ii), (iii), (iv)

(108) The number of rows in the truth table of $\sim [p \wedge (\sim q)]$ is

(1) 2 (2) 4 (3) 6 (4) 8

(109) The conditional statement $p \rightarrow q$ is equivalent to

(1) $p \vee q$ (2) $p \vee \sim q$ (3) $\sim p \vee q$ (4) $p \wedge q$

(110) Which of the following is a tautology?

(1) $p \vee q$ (2) $p \wedge q$ (3) $p \vee \sim p$ (4) $p \wedge \sim p$

(111) Which of the following is a contradiction?

(1) $p \vee q$ (2) $p \wedge q$ (3) $p \vee \sim p$ (4) $p \wedge \sim p$

(112) $p \leftrightarrow q$ is equivalent to

(1) $p \rightarrow q$ (2) $q \rightarrow p$ (3) $(p \rightarrow q) \vee (q \rightarrow p)$ (4) $(p \rightarrow q) \wedge (q \rightarrow p)$

(113) Which of the following is not a binary operation on R

(1) $a * b = ab$ (2) $a * b = a - b$

(3) $a * b = \sqrt{ab}$ (4) $a * b = \sqrt{a^2 + b^2}$

(114) A monoid becomes a group if it also satisfies the

(1) closure axiom (2) associative axiom

(3) identity axiom (4) inverse axiom

(115) Which of the following is not a group?

(1) $(Z_n , +_n)$ (2) $(Z, +)$ (3) $(Z, \cdot)$ (4) $(R, +)$

(116) In the set of integers with operation * defined by $a * b = a + b - ab$, the value of $3 * (4 * 5)$ is

(1) 25 (2) 15 (3) 10 (4) 5

(117) The order of [7] in $(Z_9 , +_9)$ is

(1) 9 (2) 6 (3) 3 (4) 1

(118) In the multiplicative group of cube root of unity, the order of w^2 is

(1) 4 (2) 3 (3) 2 (4) 1

(119) The value of $[3] +_{11} \left([5] +_{11} [6]\right)$ is

(1) [0] (2) [1] (3) [2] (4) [3]

(120) In the set of real numbers R, an operation * is defined by

$a * b = \sqrt{a^2 + b^2}$. Then the value of (3 * 4) * 5 is

(1) 5 (2) $5\sqrt{2}$ (3) 25 (4) 50

(121) Which of the following is correct?

(1) An element of a group can have more than one inverse.

(2) If every element of a group is its own inverse, then the group is abelian.

(3) The set of all 2 × 2 real matrices forms a group under matrix multiplication.

(4) $(a * b)^{-1} = a^{-1} * b^{-1}$ for all $a, b \in G$

(122) The order of $-i$ in the multiplicative group of 4th roots of unity is

(1) 4 (ii) 3 (3) 2 (4) 1

(123) In the multiplicative group of nth roots of unity, the inverse of ω^k is $(k < n)$

(1) $\omega^{1/k}$ (2) ω^{-1} (3) ω^{n-k} (4) $\omega^{n/k}$

(124) In the set of integers under the operation * defined by $a * b = a + b - 1$, the identity element is

(1) 0 (2) 1 (3) a (4) b

(125) If $f(x) = \begin{cases} k\,x^2, & 0 < x < 3 \\ 0 & \text{,elsewhere} \end{cases}$ is a probability density function then the value of k is

(1) $\frac{1}{3}$ (2) $\frac{1}{6}$ (3) $\frac{1}{9}$ (4) $\frac{1}{12}$

(126) If $f(x) = \frac{A}{\pi} \frac{1}{16 + x^2}, -\infty < x < \infty$

is a p.d.f of a continuous random variable X, then the value of A is

(1) 16 (2) 8 (3) 4 (4) 1

(127) A random variable X has the following probability distribution

X	0	1	2	3	4	5
P(X = x)	1/4	$2a$	$3a$	$4a$	$5a$	1/4

Then $P(1 \le x \le 4)$ is

(1) $\frac{10}{21}$ (2) $\frac{2}{7}$ (3) $\frac{1}{14}$ (4) $\frac{1}{2}$

(128) A random variable X has the following probability mass function as follows :

X	−2	3	1
P(X = x)	$\frac{\lambda}{6}$	$\frac{\lambda}{4}$	$\frac{\lambda}{12}$

Then the value of λ is

(1) 1 (2) 2 (3) 3 (4) 4

(129) X is a discrete random variable which takes the values 0, 1, 2 and $P(X = 0) = \frac{144}{169}$, $P(X = 1) = \frac{1}{169}$ then the value of $P(X = 2)$ is

(1) $\frac{145}{169}$ (2) $\frac{24}{169}$ (3) $\frac{2}{169}$ (4) $\frac{143}{169}$

(130) A random variable X has the following p.d.f

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

The value of k is

(1) $\frac{1}{8}$ (2) $\frac{1}{10}$ (3) 0 (4) -1 or $\frac{1}{10}$

(131) Given $E(X + c) = 8$ and $E(X - c) = 12$ then the value of c is

(1) −2 (2) 4 (3) −4 (4) 2

(132) X is a random variable taking the values 3, 4 and 12 with probabilities $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{5}{12}$. Then $E(X)$ is

(1) 5 (2) 7 (3) 6 (4) 3

(133) Variance of the random variable X is 4. Its mean is 2. Then $E(X^2)$ is

(1) 2 (2) 4 (3) 6 (4) 8

(134) $\mu_2 = 20$, $\mu_2' = 276$ for a discrete random variable X. Then the mean of the random variable X is

(1) 16 (2) 5 (3) 2 (4) 1

(135) Var $(4X + 3)$ is

(1) 7 (2) 16 Var (X) (3) 19 (4) 0

(136) In 5 throws of a die, getting 1 or 2 is a success. The mean number of successes is

(1) $\frac{5}{3}$ (2) $\frac{3}{5}$ (3) $\frac{5}{9}$ (4) $\frac{9}{5}$

(137) The mean of a binomial distribution is 5 and its standard deviation is 2. Then the value of n and p are

(1) $\left(\frac{4}{5}, 25\right)$ (2) $\left(25, \frac{4}{5}\right)$ (3) $\left(\frac{1}{5}, 25\right)$ (4) $\left(25, \frac{1}{5}\right)$

(138) If the mean and standard deviation of a binomial distribution are 12 and 2 respectively. Then the value of its parameter p is

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$ (3) $\frac{2}{3}$ (4) $\frac{1}{4}$

(139) In 16 throws of a die getting an even number is considered a success. Then the variance of the successes is

(1) 4 (2) 6 (3) 2 (4) 256

(140) A box contains 6 red and 4 white balls. If 3 balls are drawn at random, the probability of getting 2 white balls without replacement, is

(1) $\frac{1}{20}$ (2) $\frac{18}{125}$ (3) $\frac{4}{25}$ (4) $\frac{3}{10}$

(141) If 2 cards are drawn from a well shuffled pack of 52 cards, the probability that they are of the same colours without replacement, is

(1) $\frac{1}{2}$ (2) $\frac{26}{51}$ (3) $\frac{25}{51}$ (4) $\frac{25}{102}$

(142) If in a Poisson distribution $P(X = 0) = k$ then the variance is

(1) $\log \frac{1}{k}$ (2) $\log k$ (3) e^{λ} (4) $\frac{1}{k}$

(143) If a random variable X follows Poisson distribution such that $E(X^2) = 30$ then the variance of the distribution is

(1) 6 (2) 5 (3) 30 (4) 25

(144) The distribution function $F(X)$ of a random variable X is

(1) a decreasing function

(2) a non-decreasing function

(3) a constant function

(4) increasing first and then decreasing

(145) For a Poisson distribution with parameter $\lambda = 0.25$ the value of the 2^{nd} moment about the origin is

(1) 0.25 (2) 0.3125 (3) 0.0625 (4) 0.025

(146) In a Poisson distribution if $P(X = 2) = P(X = 3)$ then the value of its parameter λ is

(1) 6 (2) 2 (3) 3 (4) 0

(147) If $f(x)$ is a p.d.f of a normal distribution with mean μ then $\int_{-\infty}^{\infty} f(x)\, dx$ is

(1) 1 (2) 0.5 (3) 0 (4) 0.25

(148) The random variable X follows normal distribution

$f(x) = ce^{\frac{-1/2\,(x-100)^2}{25}}$ Then the value of c is

(1) $\sqrt{2\pi}$ (2) $\frac{1}{\sqrt{2\pi}}$ (3) $5\sqrt{2\pi}$ (4) $\frac{1}{5\sqrt{2\pi}}$

(149) If $f(x)$ is a *p.d.f.* of a normal variate X and $X \sim N(\mu, \sigma^2)$ then $\int_{-\infty}^{\mu} f(x)\, dx$ is

(1) undefined (2) 1 (3) .5 (4) – .5

(150) The marks secured by 400 students in a Mathematics test were normally distributed with mean 65. If 120 students got more marks above 85, the number of students securing marks between 45 and 65 is

(1) 120 (2) 20 (3) 80 (4) 160

ANSWERS

EXERCISE 5.1

(1) (i) 100 m / sec (ii) $t = 4$ (iii) 200m (iv) – 100 m / sec

(2) – 12, 0 (3) (i) 72 km / hr (ii) 60 m

(4) 1.5936°c / sec (5) decreasing at the rate of 1.6 cm / min

(6) $\frac{195}{\sqrt{29}}$ km / hr (7) 0.3 m^2 / sec (8) $\frac{\pi}{\sqrt{63}}$ m / min (9) $\frac{6}{5\pi}$ ft / min

EXERCISE 5.2

(1) (i) $8x + y + 9 = 0$ (ii) $2x - y - \pi/2 = 0$

$x - 8y + 58 = 0$ $x + 2y - 3\pi/2 = 0$

(iii) $y = 2$

$x = \pi/6$

(iv) $y - (\sqrt{2} + 1) = (2 + \sqrt{2})\left(x - \frac{\pi}{4}\right)$

$y - (\sqrt{2} + 1) = \frac{-1}{2 + \sqrt{2}}\left(x - \frac{\pi}{4}\right)$

(2) $\left(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ and $\left(-2\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$

(3) (2, 3) and (– 2, – 3) (4) (i) (1, 0) and (1, 4) (ii) (3, 2) and (–1, 2)

(5) $2x + 3y \pm 26 = 0$ (6) $x + 9y \pm 20 = 0$

(9) $\theta = \tan^{-1}\left[\left|\frac{\log a - \log b}{1 + \log a \ \log b}\right|\right]$

EXERCISE 5.3

(1) (i) True , $c = \frac{\pi}{2}$ (ii) Fails , $f(0) \neq f(1)$

(iii) Fails , At $x = 1$ the function is not differentiable (iv) True , $c = \pm\frac{\sqrt{3}}{2}$

(2) (0, 1)

EXERCISE 5.4

(1) (i) True , $c = \frac{3}{2}$ (ii) True , $c = \sqrt{2}$

(iii) True , $c = \frac{-1 + \sqrt{61}}{6}$

(iv) Fails , Function is not differentiable at $x = 0$ (v) True , $c = \frac{7}{3}$

(2) 16

EXERCISE 5.5

(1) $1 + \frac{2x}{\lfloor 1} + \frac{(2x)^2}{\lfloor 2} + \frac{(2x)^3}{\lfloor 3} + \ldots$ (2) $1 - x^2 + \frac{x^4}{3} + \ldots$

(3) $1 - x + x^2 + \ldots$ (4) $x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

EXERCISE 5.6

(1) $-\pi$	(2) 2	(3) 1	(4) $n2^{n-1}$	(5) 2
(6) -2	(7) 0	(8) 2	(9) 0	(10) e
(11) 1	(12) 1	(13) 1		

EXERCISE 5.7

(3) (i) increasing (ii) st. increasing (iii) st. decreasing

(iv) st. increasing (v) increasing

(5) (i) increasing in $(-\infty, -1/2\,]$ and decreasing in $[-1/2, \infty)$

(ii) increasing in $(-\infty, -1] \cup [1, \infty)$ and decreasing in $[-1, 1]$

(iii) strictly increasing on R

(iv) decreasing in $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$ increasing in $\left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

(v) increasing in $[0, \pi]$

(vi) increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and decreasing in $\left[0, \frac{\pi}{4}\right]$

EXERCISE 5.9

	Critical numbers	**Stationary points**
(1) (i)	$x = \frac{1}{3}$	$\left(\frac{1}{3}, \frac{1}{3}\right)$
(ii)	$x = \pm 1$	$(1, -1)$ and $(-1, 3)$
(iii)	$x = 0, 4, \frac{8}{7}$	$(4, 0)$ and $\left(\frac{8}{7}, \left(\frac{8}{7}\right)^{4/5}\left(\frac{20}{7}\right)^2\right)$
(iv)	$x = 0, -2$	$(0, 1)$ and $\left(-2, -\frac{1}{3}\right)$
(v)	$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$	$(0, 0)$ $\left(\frac{\pi}{4}, 1\right)$ $\left(\frac{\pi}{2}, 0\right)$ $\left(\frac{3\pi}{4}, 1\right)$ $(\pi, 0)$
(vi)	$\theta = \pi$	(π, π)

		Absolute maximum	Absolute minimum
(2)	(i)	5	1
	(ii)	2	– 7
	(iii)	66	– 15
	(iv)	3	$\sqrt{5}$
	(v)	$\frac{2}{3}$	$\frac{1}{2}$
	(vi)	$\sqrt{2}$	1
	(vii)	$\pi + 2$	$-\frac{\pi}{6} - \sqrt{3}$

		Local maximum	Local minimum
(3)	(i)	$\frac{2}{3\sqrt{3}}$	$\frac{-2}{3\sqrt{3}}$
	(ii)	12	$\frac{-19}{27}$
	(iii)	0	– 9
	(iv)	Nil	– 1
	(v)	1	Nil
	(vi)	No maximum and no minimum	

EXERCISE 5.10

(1) 50, 50 (2) 10, 10 (5) $\left(\sqrt{2}\, r, \sqrt{2}\, r\right)$ (6) $20\sqrt{5}$

EXERCISE 5.11

	Concave upward	Convex upward	Points of inflection
(1)	$(-\infty, 1)$	$(1, \infty)$	$(1, 0)$
(2)	R	–	Nil
(3)	$\left(-\frac{5}{6}, \infty\right)$	$\left(-\infty, -\frac{5}{6}\right)$	$-\frac{5}{6}, \frac{305}{54}$
(4)	$(-\infty, -1) \cup (1, \infty)$	$(-1, 1)$	$(1, -5), (-1, -5)$
(5)	$\left(\frac{\pi}{2}, \pi\right)$	$\left(0, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, 0\right)$
(6)	$(-2, 1)$	$(-\infty, -2) \cup (1, \infty)$	$(1, 9), (-2, 48)$

EXERCISE 6.1

(1) (i) $dy = 5x^4 dx$ (ii) $dy = \frac{1}{4} x^{-3/4} dx$ (iii) $dy = \frac{x(2x^2+1)}{\sqrt{x^4+x^2+1}} dx$

(iv) $dy = \left[\frac{7}{(2x+3)^2}\right] dx$ (v) $dy = 2 \cos 2x \, dx$

(vi) $dy = (x \sec^2 x + \tan x) \, dx$

(2) (i) $dy = -2x \, dx$; $dy = -5$ (ii) $dy = (4x^3 - 6x + 1) \, dx$; $dy = 2.1$

(iii) $dy = 6x(x^2+5)^2 dx$; $dy = 10.8$ (iv) $dy = -\frac{1}{2\sqrt{1-x}} dx$; $dy = -0.01$

(v) $dy = -\sin x \, dx$; $dy = -0.025$

(3) (i) 6.008 (app.) (ii) 0.099 (app.) (iii) 2.0116 (app.) (iv) 58.24 (app.)

(4) (i) 270 cubic cm (ii) 36 cm^2 (5) (i) 0.96 π cm^2 (ii) 0.001667

EXERCISE 6.2

No.	Existence	Symmetry	Asymptote	Loops
2	$-1 \le x \le 1$	Both axes and hence origin	No asymptotes	2 loops between -1 and 1
3	$-2 < x \le 6$	x-axis	$x = -2$	1 loop between 0 and 6
4	$x \le 1$	x-axis	No asymptotes	1 loop between 0 and 1
5	$x = b$ and $x \ge a$	x - axis	No asymptotes	No loops

EXERCISE 6.3

(1) (i) $\frac{\partial u}{\partial x} = 2x + 3y$; $\frac{\partial u}{\partial y} = 3x + 2y$ (ii) $\frac{\partial u}{\partial x} = \frac{x^3 + 2y^3}{x^3 y^2}$; $\frac{\partial u}{\partial y} = -\frac{(y^3 + 2x^3)}{x^2 y^3}$

$\frac{\partial^2 u}{\partial x^2} = 2$; $\frac{\partial^2 u}{\partial y^2} = 2$ $\frac{\partial^2 u}{\partial x^2} = \frac{-6y}{x^4}$; $\frac{\partial^2 u}{\partial y^2} = \frac{6x}{y^4}$

(iii) $\dfrac{\partial u}{\partial x} = 3\cos 3x \ \cos 4y$; $\dfrac{\partial u}{\partial y} = -4\sin 3x \sin 4y$

$\dfrac{\partial^2 u}{\partial x^2} = -9\sin 3x \cos 4y$; $\dfrac{\partial^2 u}{\partial y^2} = -16\sin 3x \cos 4y$

(iv) $\dfrac{\partial u}{\partial x} = \dfrac{y}{x^2+y^2}$; $\dfrac{\partial u}{\partial y} = -\dfrac{x}{x^2+y^2}$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{-2xy}{(x^2+y^2)^2}$; $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{2xy}{(x^2+y^2)^2}$

(3) (i) $5t^4e^{t^5}$ (ii) $\dfrac{2(e^{2t}-e^{-2t})}{(e^{2t}+e^{-2t})}$ (iii) $-\sin t$

(iv) $2\cos^2 t$

(4) (i) $\dfrac{\partial w}{\partial r} = \dfrac{2}{r}$; $\dfrac{\partial w}{\partial \theta} = 0$ (ii) $\dfrac{\partial w}{\partial u} = 4u(u^2+v^2)$; $\dfrac{\partial w}{\partial v} = 4v(u^2+v^2)$

(iii) $\dfrac{\partial w}{\partial u} = \dfrac{2u}{\sqrt{1-(u^2-v^2)^2}}$; $\dfrac{\partial w}{\partial v} = \dfrac{-2v}{\sqrt{1-(u^2-v^2)^2}}$

EXERCISE 7.1

(1) $\dfrac{\pi}{4}$ (2) $\dfrac{2}{3}$ (3) $\dfrac{\sqrt{5}}{2} + \dfrac{9}{4}\sin^{-1}\left(\dfrac{2}{3}\right)$ (4) $\dfrac{1}{4}$

(5) $\dfrac{\pi}{6}$ (6) $\dfrac{1}{3}\tan^{-1}\dfrac{1}{3}$ (7) $\log\left(\dfrac{16}{15}\right)$ (8) $\dfrac{1}{64}\pi^4$

(9) $\dfrac{2}{3}$ (10) $e-2$ (11) $\dfrac{1}{10}(e^{3\pi/2}-3)$ (12) $\dfrac{1}{2}[1-e^{-\pi/2}]$

EXERCISE 7.2

(1) 0 (2) 0 (3) $\dfrac{1}{4}$ (4) $\dfrac{4}{3}$

(5) $\dfrac{2}{3}$ (6) 0 (7) 0 (8) $\dfrac{3}{2}$

(9) $\dfrac{1}{132}$ (10) $\dfrac{\pi}{12}$

EXERCISE 7.3

(2) (i) $-\dfrac{1}{4}\sin^3 x\cos x - \dfrac{3}{8}\sin x\cos x + \dfrac{3}{8}x$

(ii) $\dfrac{1}{5}\cos^4 x\sin x + \dfrac{4}{15}\cos^2 x\sin x + \dfrac{8}{15}\sin x$

(3) (i) $\frac{5\pi}{32}$ (ii) $\frac{128}{315}$ (4) (i) $\frac{35\pi}{512}$ (ii) $\frac{16}{105}$ (5) (i) $\frac{-3}{4}e^{-2}+\frac{1}{4}$ (ii) $2^7.\underline{|6}$

EXERCISE 7.4

(1) (i) 4 (ii) 4 (2) (i) 57 (ii) 16 (3) 4

(4) $\frac{55}{27}$ (5) $8\left(4-\sqrt{2}\right)$ (6) $\frac{8a^2}{3}$

(7) $\frac{4\sqrt{5}}{3}\left[\sqrt{5}+\frac{9}{2}\sin^{-1}\frac{2}{3}\right]$ (8) 9 (9) 4

(10) πa^2 (11) $\frac{178\pi}{15}$ (12) $\frac{\pi a^3}{24}$

(13) $\frac{3}{5}\pi$ (14) $\frac{4\pi\, ab^2}{3}$ (15) $\frac{1}{3}\pi r^2 h$ (16) π

EXERCISE 7.5

(1) $2\pi a$ (2) $4a$ (3) $\frac{8\pi a^2}{3}\left(2\sqrt{2}-1\right)$

EXERCISE 8.1

		order	degree		order	degree
(1)	(i)	1	1	(vi)	2	3
	(ii)	1	1	(vii)	2	1
	(iii)	2	1	(viii)	2	2
	(iv)	2	2	(ix)	1	3
	(v)	3	3	(x)	1	1

(2) (i) $y = 2xy'$ (ii) $x^2y'' - 2xy' + 2y - 2c = 0$

(iii) $xy' + y = 0$ (iv) $x\left[(y')^2 + yy''\right] - yy' = 0$

(v) $y'' + 3y' - 10y = 0$ (vi) $y'' = 6y' - 9y$

(vii) $y'' = 6y' - 13y$ (viii) $y = e^{(y'/y)x}$

(ix) $y'' - 4y' + 13y = 0$

(3) (i) $yy' = (y')^2x + a$ (ii) $y' = m$ (iii) $y'' = 0$ (4) $y^2\left[(y')^2 + 1\right] = 1$

EXERCISE 8.2

(1) $y + \frac{\sin 2y}{2} + \frac{\cos 7x}{7} + \frac{\cos 3x}{3} = c$ (2) $\log y + e^{\tan x} = c$

(3) $x = cy\, e^{\left(\frac{x+y}{xy}\right)}$ (4) $e^x(x^2 - 2x + 2) + \log y = c$

(5) $\sin^{-1}\left(\frac{y-4}{5}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+5}{\sqrt{3}}\right) = c$ (6) $\tan(x+y) - \sec(x+y) = x + c$

(7) $y - \tan^{-1}(x + y) = c$ (8) $e^{xy} = x + 1$

EXERCISE 8.3

(1) $(y - 2x) = cx^2 y$ (2) $y^3 = cx^2 e^{-x/y}$ (3) $y = ce^{x^2/2y^2}$

(4) $2y = x(x + y)$ (5) $x^2 (x^2 + 4y^2)^3 = c$ (6) $y = x \log x$

EXERCISE 8.4

(1) $e^x(y - x + 1) = c$ (2) $y(x^2 + 1)^2 - x = c$

(3) $xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$ (4) $y(1 + x^2) = \sin x + c$

(5) $2xy + \cos x^2 = c$ (6) $y = 1 + ce^{-x^2/2}$

(7) $xe^y = \tan y + c$ (8) $x = y - a^2 + ce^{-y/a^2}$

EXERCISE 8.5

(1) $y = Ae^{-4x} + Be^{-3x} + \frac{e^{2x}}{30}$ (2) $y = e^{2x}[A\cos 3x + B\sin 3x] + \frac{e^{-3x}}{34}$

(3) $y = (Ax + B)e^{-7x} + \frac{x^2}{2}e^{-7x} + \frac{4}{49}$ (4) $y = Ae^{12x} + Be^x + \frac{e^{-2x}}{42} - \frac{5}{11}xe^x$

(5) $y = 2[\cos x - \sin x]$ (6) $y = e^x[2 - 3e^x + e^{2x}]$

(7) $y = Ae^x + Be^{-4x} - \frac{1}{4}\left[x^2 + \frac{3x}{2} + \frac{13}{8}\right]$

(8) $y = Ae^{3x} + Be^{-x} + \frac{1}{130}\ [4\cos 2x - 7\sin 2x]$

(9) $y = (A + Bx) + \sin 3x$ (10) $y = (A + Bx)e^{3x} + \left(\frac{x}{9} + \frac{2}{27}\right) + e^{2x}$

(11) $y = Ae^x + Be^{-x} - \frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x$

(12) $y = \left[C\cos\sqrt{5}\,x + D\sin\sqrt{5}\,x\right] + \frac{1}{10} + \frac{1}{2}\cos 2x$

(13) $y = e^{-x}\left[C\cos\sqrt{2}\,x + D\sin\sqrt{2}\,x\right] - \frac{1}{17}[4\cos 2x + \sin 2x]$

(14) $y = Ae^{-x} + Be^{-x/3} + \frac{3}{2}xe^{-x/3}$

EXERCISE 8.6

(1) $A = 0.9025\,A_0$ (2) 17 years (app.) (3) 38.82° C

(4) 197600 (5) 136 days

EXERCISE 9.1

Statements : (1), (2), (3), (5), (6), (10) ; others are not statements.

(11) *T*	(12) *T*	(13) *T*	(14) *F*	(15) *T*
(16) *F*	(17) *F*	(18) *T*	(19) *F*	(20) *F*

(21) (i) Anand reads newspaper and plays cricket

Anand reads newspaper or plays cricket.

(ii) I like tea and ice-cream

I like tea or ice-cream

(22) (i) $p \vee q$: Kamala is going to school or there are twenty students in the class.

(ii) $p \wedge q$: Kamala is going to school and there are twenty students in the class.

(iii) Kamala is not going to school.

(iv) It is false that there are twenty students in the class.

(v) Kamala is not going to school or there are twenty students in the class.

(23) (i) $p \wedge q$ (ii) $p \vee q$ (iii) $\sim p$ (iv) $p \wedge q$ (v) $\sim p$

(24) Sita likes neither reading nor playing.

(25) (i) $\sqrt{5}$ is not an irrational number.

(ii) Mani is not sincere or not hardworking.

(iii) This picture is nither good nor beautiful.

EXERCISE 9.2

(1) Truth table for $pv\ (\sim q)$

p	q	$\sim q$	$p \vee (\sim q)$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(2) Truth table for $(\sim p) \wedge (\sim q)$

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(3) Truth table for $\sim (p \vee q)$

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(4) Truth table for $(p \vee q) \vee (\sim p)$

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee (\sim p)$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

(5) Truth table for $(p \wedge q) \vee (\sim q)$

p	q	$p \wedge q$	$\sim q$	$(p \wedge q) \vee (\sim q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	T	T

(6) Truth table for $\sim (p \vee (\sim q))$

p	q	$\sim q$	$p \vee (\sim q)$	$\sim (p \vee (\sim q))$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

(7) Truth table for $(p \wedge q) \vee (\sim (p \wedge q))$

p	q	$p \wedge q$	$\sim (p \wedge q)$	$(p \wedge q) \vee (\sim (p \wedge q))$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(8) Truth table for $(p \wedge q) \wedge (\sim q)$

p	q	$p \wedge q$	$\sim q$	$(p \wedge q) \wedge (\sim q)$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	F
F	F	F	T	F

(9) Truth table for $(p \vee q) \vee r$

p	q	r	$p \vee q$	$(p \vee q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

(10) Truth table for $(p \wedge q) \vee r$

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

EXERCISE 9.3

(1) (i) $((\sim p) \wedge q) \wedge p$ contradiction

(ii) $(p \vee q) \vee (\sim (p \vee q))$ Tautology

(iii) $(p \wedge (\sim q)) \vee ((\sim p) \vee q)$ Tautology

(iv) $q \vee (p \vee (\sim q))$ Tautology

(v) $(p \wedge (\sim p)) \wedge ((\sim q) \wedge p))$ Contradiction

EXERCISE 9.4

(1) Non-commutative but associative (2) Yes, Identity element is 1

(10) $0([1]) = 1$, $0([2]) = 4$, $0([3]) = 4$, $0([4]) = 2$

EXERCISE 10.1

(1)

X	0	1	2	3
$p(X = x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

(2)

X	0	1	2
$p(X = x)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

(3)

X	0	1	2
$p(X = x)$	$\frac{12}{22}$	$\frac{9}{22}$	$\frac{1}{22}$

(4) (i) $\frac{1}{81}$ (ii) $\frac{1}{9}$ (iii) $\frac{11}{27}$ (6) (i) 20 (ii) $\frac{13}{16}$

(7) (i) $\alpha\,\beta$ (ii) $e^{-\beta(10^{\alpha})}$

(8) $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$ (i) 0.3125 (ii) 0.25 (iii) 0.4375

(9) $c = a$ (10) (i) $\frac{1}{2\pi}$ (ii) $\frac{1}{4}$ (iii) $\frac{1}{2}$

EXERCISE 10.2

(1) Mean = 1, Variance = $\frac{1}{2}$ (2) $E(X) = 3.5$

(3) $E(X) = -15$ (4) Mean = $\frac{2}{13}$, Variance = $\frac{24}{169}$

(5) $E(X) = -1.25$

(6) Mean = 6.4 , Variance = 16.24

(7) (i) Mean = 0, Variance = 48

(ii) Mean = $\frac{1}{\alpha}$, Variance = $\frac{1}{\alpha^2}$

(iii) Mean = 2, Variance = 2

EXERCISE 10.3

(1) Not possible as probability of an event can lie between 0 and 1 only.

(2) Mean = 40 ; Variance = $\frac{80}{3}$

(3) Mean = 450, standard deviation = $3\sqrt{5}$

(4) (i) $\frac{3}{8}$ (ii) $\frac{11}{16}$ (iii) $\frac{11}{16}$ (5) $\frac{2048}{5^5}$ (6) $\frac{5^9}{6^{10}}$ (15)

EXERCISE 10.4

(1) (i) 0.4331 (ii) 0.5368 (2) (i) 0.1952 (ii) 0.5669

(3) (i) $45 \times \frac{4^8}{5^{10}}$ (ii) 0.2706 (4) (i) 0.0838 (ii) 0.9598

(5) (i) approximately 50 drivers (ii) approximately 353 drivers

EXERCISE 10.5

(1) (i) 0.9772 (ii) 0.5 (iii) 0.9104

(iv) 0.8413 (v) 0.2417

(2) (i) 0.67 (ii) – 0.52 and 0.52 (iii) – 1.04

(3) 0.0749 (4) 4886 pairs

(5) (i) 291 persons (app.) (ii) 6 persons (app.)

(6) 72.19 inches (7) 640 students (8) $c = \frac{e^{-9/4}}{\sqrt{\pi}}$, $\mu = \frac{3}{2}$, $\sigma^2 = \frac{1}{2}$

KEY TO OBJECTIVE TYPE QUESTIONS

Q.No	Key	Q.No	Key	Q.No	Key	Q.No	Key	Q.No	Key
1	4	31	2	61	2	91	2	121	2
2	2	32	4	62	2	92	4	122	1
3	3	33	1	63	1	93	2	123	3
4	2	34	1	64	4	94	3	124	2
5	1	35	3	65	2	95	2	125	3
6	3	36	2	66	4	96	1	126	3
7	4	37	1	67	1	97	2	127	4
8	2	38	2	68	2	98	3	128	2
9	3	39	3	69	3	99	4	129	2
10	4	40	4	70	2	100	3	130	2
11	1	41	4	71	2	101	1	131	1
12	2	42	1	72	3	102	2	132	2
13	2	43	4	73	3	103	3	133	4
14	1	44	3	74	4	104	4	134	1
15	1	45	1	75	4	105	4	135	2
16	4	46	3	76	1	106	1	136	1
17	2	47	4	77	1	107	3	137	4
18	1	48	2	78	1	108	2	138	3
19	1	49	3	79	2	109	3	139	1
20	2	50	1	80	4	110	3	140	4
21	3	51	1	81	3	111	4	141	3
22	1	52	2	82	2	112	4	142	1
23	1	53	3	83	2	113	3	143	2
24	2	54	2	84	1	114	4	144	2
25	2	55	3	85	2	115	3	145	2
26	3	56	1	86	2	116	1	146	3
27	2	57	1	87	3	117	1	147	1
28	4	58	4	88	4	118	2	148	4
29	2	59	4	89	2	119	4	149	3
30	1	60	2	90	1	120	2	150	3

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